Resonances

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Joint work with F. Naud (Avignon)

• [JN1] *Lower bounds for resonances of infinite area Riemann surfaces*.
Journal of Analysis and PDE, vol. 3 (2010), no. 2, 207-225.

• [JN2] *On the resonances of convex co-compact subgroups of arithmetic groups*
arXiv:1011.6264

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Let Γ be a *geometrically finite* subgroup of $\mathrm{PSL}(2, \mathbf{R})$ without elliptic elements. Fundamental domain $X = \Gamma\backslash\mathbf{H}^2$ has finitely many sides. Assume that X has *infinite* hyperbolic area: X decomposes into a finite area surface N (called *Nielsen region* or *convex core*) to which finitely many infinite area half-cylinders (*funnels*) are glued.
If Γ has parabolic elements, then N has *cusps* (parabolic vertices); a surface without cusps is called *convex co-compact*; then Γ has no parabolic elements.

- The spectrum of $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on X consists of the continuous spectrum $[1/4, +\infty]$ (no embedded eigenvalues).
- δ is the Hausdorff dimension of the limit set $\Lambda(\Gamma) \subset S^1$. If $\delta > 1/2$, Δ has finitely many eigenvalues in $(0, 1/4)$; the first nonzero eigenvalue $\lambda_0 = \delta(1-\delta)$. Point spectrum is empty if $\delta \leq 1/2$ (Lax, Phillips, Patterson, Sullivan).
- The resolvent

$$R(\lambda) = \left(\Delta_X - \frac{1}{4} - \lambda^2\right)^{-1} : L^2(X) \to L^2(X)$$

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- *Resonances* are the poles of meromorphic continuation of the resolvent $R(\lambda) : C_0^\infty(X) \to C^\infty(X)$ to the whole complex plane **C**. Their set is denoted by $\mathcal{R}_X$. Guillopé and Zworski showed that $\exists C > 0$ such that

$$1/C < \#\{z \in \mathcal{R}_X : |z| < R\}/R^2 < C, \qquad R \to \infty.$$

- Finer asymptotics: let

$$N_C(T) = \#\{z \in \mathcal{R}_X : \Im(z) \le C, |\Re(z)| \le T\}.$$

- Zworski, Guillopé and Lin: "fractal" upper bound **Theorem 1.** For convex co-compact X, $N_C(T) = O(T^{1+\delta})$; where C is fixed, and $T \to \infty$. They conjectured the upper bound is sharp.

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- **Lower bounds:** Guillopé, Zworski: $\forall \epsilon > 0, \ \exists C_\epsilon > 0$, such that

$$N_{C_\epsilon}(T) = \Omega(T^{1-\epsilon}).$$

  The proof uses a wave trace formula for resonances on X and takes into account contributions from a *single* closed geodesic on X.
- **Question:** Can one improve lower bounds taking into account contributions from *many* closed geodesics on X?
- **Answer:** Yes, this is done in [JN1].

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- Guillopé, Lin, Zworski: let

$$\mathcal{D}(z) = \{\lambda \in \mathcal{R}_X : |\lambda - z| \leq 1\}$$

Then for all $z : \Im(z) \leq C$, we have $\mathcal{D}(z) = O(|\Re(z)|^{\delta})$.

- Let $A > 0$, and let W_A denote the logarithmic neighborhood of the real axis:

$$W_A = \{\lambda \in \mathbf{C} : \Im\lambda \leq A\log(1 + |\Re\lambda|)\}$$

- **Theorem 2.** Let X be a geometrically finite hyperbolic surface of infinite area, and let $\delta > 1/2$. Then there exists a sequence $\{z_i\} \in W_A, \Re(z_i) \to \infty$ such that

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- **Corollary:** If $\delta > 1/2$, then $W_A \cap \mathcal{R}_X$ is different from a lattice.
- Examples of Γ such that $\delta(\Gamma) > 1/2$ are easy to construct. Pignataro, Sullivan: fix the topology of X. Denote by $l(X)$ the maximum length of the closed geodesics that form the boundary of N. Then $\lambda_0(X) \leq C(X)l(X)$, where $C = C(X)$ depends only on the topology of X. By Patterson-Sullivan, $\lambda_0 < 1/4 \Leftrightarrow \delta > 1/2$, so letting $l(X) \to 0$ gives many examples.
- Proof of Theorem 2 uses (a version of Selberg) trace formula due to Guillopé and Zworski, and Dirichlet box principle.

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- Theorem 2 gives a *logarithmic* lower bound $\mathcal{D}(z_i) \geq (\log|\Re(z_i)|)^{\frac{\delta-1/2}{\delta}-\epsilon}$ for an infinite sequence of disks $D(z_i, 1)$. Conjecture of Guillopé and Zworski would imply that $\forall\epsilon > 0\ \exists\{z_i\}$ such that $\mathcal{D}(z_i) \geq |\Re(z_i)|^{\delta-\epsilon}$.
- **Question:** can one get *polynomial* lower bounds for some particular groups Γ?
  Answer: Yes. **Idea:** look at infinite index subgroups of arithmetic groups, and use methods of Selberg-Hejhal.
- **Theorem 3.** Let Γ be an infinite index geom. finite subgroup of an arithmetic group Γ_0 derived from a quaternion algebra. Let $\delta(\Gamma) > 3/4$. Then $\forall\epsilon > 0, \forall A > 0$, there exists $\{z_i\} \subset W_A, \Re(z_i) \to \infty$, such that

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Key ideas:

- Number of closed geodesics on X:

$$\#\{\gamma : l(\gamma) < T\} \sim \frac{e^{\delta T}}{\delta T}, \qquad T \to \infty.$$

- Number of *distinct* closed geodesics in the arithmetic case: for Γ derived from a quaternion algebra, one has

$$\#\{L < T : L = l(\gamma)\} \ll e^{T/2}.$$

  Accordingly, for $\delta > 1/2$, there exists *exponentially large* multiplicities in the length spectrum.

- Distinct lengths are well-separated in the arithmetic case: for $l_1 \neq l_2$, we have

$$|l_1 - l_2| \gg e^{-\max(l_1, l_2)/2}.$$

  Ex: $M_1, M_2 \in \mathrm{SL}(2, \mathbf{Z}), \mathrm{tr} M_1 \neq \mathrm{tr} M_2$ then $|\mathrm{tr} M_1 - \mathrm{tr} M_2| = 2|\cosh(l_1/2) - \cosh(l_2/2)| \geq 1.$

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Trace formula (Guillopé, Zworski): Let $\psi \in C_0^\infty((0,+\infty))$, and N - Nielsen region. Then (in case there are no cusps)

$$\sum_{\lambda \in \mathcal{R}_X} \widehat{\psi}(\lambda) = -\frac{V(N)}{4\pi} \int_0^{+\infty} \frac{\cosh(t/2)}{\sin^2(t/2)} \psi(t) dt + \sum_{\gamma \in \mathcal{P}} \sum_{k \geq 1} \frac{l(\gamma)\psi(kl(\gamma))}{2\sinh(kl(\gamma)/2)},$$

where $\mathcal{P} = \{\textit{primitive closed geodesics on } X\}$.
For $\alpha, t \gg 0$, we take

$$\psi_{t,\alpha}(x) = e^{-itx}\psi_0(x-\alpha),$$

where $\psi_0 \in C_0^\infty([-1,1]), \psi \geq 0$, and $\psi_0 = 1$ on $[-1/2, 1/2]$.

- Geometric side (sum over closed geodesics):

$$S_{\alpha,t} = \sum_{\alpha-1 \leq kl(\gamma) \leq \alpha+1} \frac{l(\gamma)\psi_0(kl(\gamma)-\alpha)}{2\sinh(kl(\gamma)/2)} e^{-itkl(\gamma)}.$$

- **Lemma 4:** $\exists A > 0$ s.t. $\forall T > 0$, if we let $\alpha = 2\log T - A$, and

$$J(T) = \int_T^{3T} \left(1 - \frac{|t-2T|}{T}\right) |S_{\alpha,t}|^2 dt,$$

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Lemma 4 $\Rightarrow$ Theorem 3: Assume for contradiction that for all $z \in W_A, \Re(z) \geq R_0$ we have $\mathcal{D}(z) \leq |\Re(z)|^\beta$. Let $\alpha = 2\log T - A$. We have

$$\frac{C_2 T^{1+4\delta-3}}{(\log T)^2} \leq J(T) \leq \int_T^{3T} |S_{\alpha,T}|^2 dt.$$

Assumption implies that

$$S_{\alpha,T} = O(1 + t^\beta + T^{2\delta-3}).$$

Integrating, we find that

$$J(T) = O(T^{2\beta+1}).$$

This leads to a contradiction if $2\beta + 1 < 1 + 4\delta - 3$, or $\beta < 2\delta - 3/2$, proving Theorem 3.

Proof of Lemma 4 uses the fact that geodesic lengths on X have exponentially high multiplicities and their lengths are well-separated.
After expanding $|S^2_{\alpha,T}|^2$ and integrating, we write $J(T) = J_1(T) + J_2(T)$, where $J_1(T)$ is the *diagonal* term

$$J_1(T) = T \sum_{l \in \mathcal{L}(\Gamma)} \frac{(l^{\#}\mu(l))^2 \psi_0^2(l-\alpha)}{4\sinh^2(l/2)},$$

where $\mathcal{L}_\Gamma$ denotes set of distinct lengths of closed geodesics on X; $\mu(l)$ is the multiplicity of l; $l^{\#}$ the primitive length of a closed geodesic.
$J_1(T) \geq 0$, and $J_2(T)$ denotes the off-diagonal term. $J_2(T)$ involves integrals $\int_T^{3T}(1-|t-2T|/T)e^{i(l_1-l_2)t}dt$, where $l_1 \leq l_2$. Since distinct l_j-s are well-separated, we get cancellation in $J_2(T)$. One can show that $|J_2(T)| \leq J_1(T)/2$ with α, T chosen as in Lemma.

- It remains to bound $J_1(T)$ from below. $\psi_0(l-\alpha)$ is supported on $[\alpha-1,\alpha+1]$. The denominator $4\sinh^2(l/2)$ is of order e^α. We find that

$$J_1(T) \geq C_3 T e^{-\alpha} \sum_{l\in\mathcal{L}_\Gamma\cap[\alpha-1/2,\alpha+1/2]} (\mu(l))^2.$$

- Call the last sum S. Then

$$S \geq \frac{\left(\sum_{l\in\mathcal{L}_\Gamma\cap[\alpha-1/2,\alpha+1/2]}\mu(l)\right)^2}{\left(\sum_{l\in\mathcal{L}_\Gamma\cap[\alpha-1/2,\alpha+1/2]}1\right)}$$

The numerator is $\gg [e^{\delta\alpha}/\alpha]^2$ by the prime geodesic theorem. The denominator is $O(e^{\alpha/2})$ (since the lengths are well-separated). Hence $S \gg e^{(2\delta-1/2)\alpha}/\alpha^2$. Substituting $J(T) \gg S\cdot T/e^\alpha$, $\alpha = 2\log T - A$, we get $J(T) \gg T^{4\delta-2}/(\log T)^2$, proving Lemma 4.

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It follows from a recent result of Lewis Bowen that in every co-finite or co-compact arithmetic Fuchsian group, one can find infinite index convex co-compact subgroups with δ arbitrarily close to 1 (and in particular $> 3/4$). A. Gamburd considered infinite index subgroups of $\mathrm{SL}_2(\mathbf{Z})$ and constructed subgroups Λ_N such that $\delta(\Lambda_N) \to 1$ as $N \to \infty$. It was shown in [JN1] that subgroups Γ_N of Λ_N (of index two) provide examples of “arithmetic” groups with $\delta(\Gamma_N) > 3/4$ for large enough N. Related questions were also considered by Bourgain and Kantorovich.
The results of [JN1] can also be generalized to hyperbolic 3-manifolds (work in progress).

We describe some results in [JN2]. Let $\lambda = s(1-s), s \in \mathbf{C}$.

- If X has finite area, then all resonances lie in the strip $0 < \Re(s) < 1/2$.
- If X has infinite area, then resonances are spread all over the half plane $\{\Re(s) < 1/2\}$. We study resonances with the *largest real part.*
- If $\delta > 1/2$, then all but finitely many resonances lie in the plane $\{\Re(s) < 1/2\}$. If $\delta \leq 1/2$, F. Naud showed that there exists $\epsilon > 0$ such that

$$\mathcal{R}_X \cap \{\Re(s) \geq \delta - \epsilon\} = \{\delta\}.$$

  Constant ϵ is not effective (follows from a Dolgopyat type estimate).
- We want to find "essential spectral gap"

$$G(\Gamma) := \inf \{\sigma < \delta \ : \ \{\Re(s) \geq \sigma\} \cap \mathcal{R}_X \text{ is finite}\}.$$

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- **Conjecture:** $G(\Gamma) = \delta/2$ (known for finite volume X, Selberg).
- **Theorem 5:** Γ - convex co-compact Fuchsian group.
- If $0 < \delta \leq 1/2$, then we have $G(\Gamma) \geq \frac{\delta(1-2\delta)}{2}$.
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We next discuss connections to lattice point counting problems.

- Lax-Phillips: Let $\delta > 1/2$. Given $z, z' \in \mathbf{H}$, let $N(T; z, z') := \#\{\gamma \in \Gamma \ : \ \mathrm{d}(z, \gamma z') \leq T\}$. Then

$$N(T; z, z') = \sum_j C_j(z, z') e^{\delta_j T} + O\left(T^{5/6} e^{(\delta+1)T/3}\right),$$

  where $\delta_j \in (1/2, \delta], \delta_j(1-\delta_j) = \lambda_j \in [0, 1/4], \delta_0 = \delta$.
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Theorem 6: Γ convex co-compact subgroup of an arithmetic group with $\delta > 1/2$. There exists a full measure subset $\mathcal{G} \subset \mathbf{H} \times \mathbf{H}$ such that for all $(z, z') \in \mathcal{G}$ and all finite expansion of the form $\sum_j Q_j(T; z, z')e^{\delta_j T}$, where $\delta_j \in \mathbf{C}$ and $Q_j(T; z, z') \in \mathbf{C}[T]$, then for all $\epsilon > 0$

$$\left| N(T; z, z') - \sum_j Q_j(T; z, z')e^{\delta_j T} \right| = \Omega\left(e^{(\delta/2 - 1/4 - \epsilon)T}\right).$$

Here $\Omega(\bullet)$ means not a $O(\bullet)$.

Approximate trace formula: let $\varphi \in C_0^\infty(\mathbf{R})$. Let

$$\psi(s) := \int_{-\infty}^{+\infty} e^{su}\varphi(u)du = \widehat{\varphi}(is),$$

where $\widehat{\varphi}$ F.T. of φ.
Proposition 7. Let $\rho < \delta$, and assume

$$\#\mathcal{F}_\rho := \#\mathcal{R}_X \cap \{\Re(s) > \rho\} < \infty.$$

Then $\forall \varepsilon > 0$ small enough, $\exists \varepsilon \leq \widetilde{\varepsilon} \leq 2\varepsilon$ s.t.

$$\sum_{k\in\mathbb{N}_0}\sum_{\gamma\in\mathcal{P}} \frac{l(\gamma)}{1-e^{-kl(\gamma)}}\varphi(kl(\gamma)) = \sum_{\lambda\in\mathcal{F}_\rho}\psi(\lambda)$$

$$+O\left(\int_{-\infty}^{+\infty}(1+|x|)^\delta|\psi(\rho+\widetilde{\varepsilon}+ix)|dx\right).$$

Here $\mathcal{P}$ - primitive closed geodesics. Constant depends on ε, ρ and Γ.

Lower bound on multiplicities for arithmetic groups: For all $\ell \in \mathcal{L}_\Gamma$, let

$$m(\ell) := \#\{(k,\gamma) \in \mathbb{N}_0 \times \mathcal{P} \ : \ \ell = kl(\gamma)\}.$$

Lemma 8. Assume $\delta(\Gamma) > 1/2$. Then $\exists A_\Gamma > 0$ such that for all T large, we have

$$\sum_{\substack{T-1 \leq \ell \leq T+1 \\ \ell \in \mathcal{L}_\Gamma}} m^2(\ell) \geq A_\Gamma \frac{e^{(2\delta - 1/2)T}}{T^2}.$$

Proof uses *bounded clustering* property of Γ (Luo, Sarnak).

Proof of Theorem 5. Test functions: Let $\xi \in \mathbf{R}$, $T \gg 0$, and

$$\varphi_{\xi,T}(x) := e^{-i\xi x}\varphi(x - T),$$

where

$$\varphi \in C_0^\infty([-2,2]); \varphi \geq 0; \varphi(x) = 1, \qquad x \in [-1,1].$$

Let $A \geq \Re(s) \geq 0$, then

$$\psi_{\xi,T}(s) := \widehat{\varphi_{\xi,T}}(is) = e^{-i\xi T} e^{sT} \widehat{\varphi}(\xi + is).$$

Let

$$\mathcal{S}_{\xi,T} := \sum_{k\in\mathbb{N}_0}\sum_{\gamma\in\mathcal{P}} \frac{l(\gamma)}{1-e^{-kl(\gamma)}} e^{-i\xi kl(\gamma)}\varphi(kl(\gamma)-T).$$

Approximate trace formula (Prop. 7) implies

$$\mathcal{S}_{\xi,T} = \sum_{\lambda\in\mathcal{F}_\rho} \psi_{\xi,T}(\lambda) + E(\xi,T),$$

where

$$E(\xi,T) = O\left(e^{(\rho+\widetilde{\varepsilon})T}|\xi|^{\delta}\right).$$

Consider

$$\mathcal{G}(\sigma, T) := \sqrt{\sigma} \int_{-\infty}^{+\infty} e^{-\sigma\xi^2} |\mathcal{S}_{\xi,T}|^2 d\xi,$$

where $\sigma = \sigma(T) > 0$ - small. One can show that

$$\mathcal{G}(\sigma, T) = \sqrt{\pi} \sum_{\ell,\ell' \in \mathcal{L}_\Lambda} a_{\ell,\ell'} \varphi(\ell - T)\varphi(\ell' - T) e^{-\frac{(\ell-\ell')^2}{4\sigma}},$$

where

$$a_{\ell,\ell'} := \frac{\widetilde{\ell}\widetilde{\ell'} m(\ell) m(\ell')}{(1 - e^{-\ell})(1 - e^{-\ell'})}.$$

It follows that

$$\mathcal{G}(\sigma, T) \geq C \sum_{\substack{T-1 \leq \ell \leq T+1 \\ \ell \in \mathcal{L}_\Gamma}} m^2(\ell).$$

Proposition 7 allows to bound $\frac{\mathcal{G}(\sigma,T)}{2\sqrt{\sigma}}$ by

$$\underbrace{\int_{-\infty}^{+\infty} e^{-\sigma\xi^2} \left| \sum_{\lambda\in\mathcal{F}_\rho} \psi_{\xi,T}(\lambda) \right|^2 d\xi}_{\mathcal{I}_1(\sigma,T)} + \underbrace{\int_{-\infty}^{+\infty} e^{-\sigma\xi^2} |E(\xi,T)|^2 d\xi}_{\mathcal{I}_2(\sigma,T)} .$$

We assume that $\mathcal{F}_\rho$ is finite, hence

$$\mathcal{I}_1(\sigma, T) = O\left(e^{2\delta T}\right),$$

uniformly in σ. Also, one can show

$$\mathcal{I}_2(\sigma, T) = O\left(e^{2(\rho+\widetilde{\varepsilon})T}\sigma^{-\delta-1/2}\right).$$

Concluding the proof: $\delta \in (0, 1/2)$**:** Cannot use Lemma 8. Use

$$\mathcal{G}(\sigma, T) \geq C \sum_{\substack{T-1 \leq \ell \leq T+1 \\ \ell \in \mathcal{L}_\Gamma}} m(\ell) \geq B \frac{e^{\delta T}}{T},$$

(using prime geodesic theorem), for $B > 0$. Let $T \gg 0, \sigma \ll 1$.

$$B \frac{e^{\delta T}}{T} = O\left(\sqrt{\sigma} e^{2\delta T}\right) + O\left(e^{2(\rho + \widetilde{\varepsilon})T} \sigma^{-\delta}\right).$$

Let $\sigma = e^{-\alpha T}$; get a contradiction as $T \to +\infty$ if

$$\alpha > 2\delta \text{ and } \rho < \frac{\delta(1-\alpha)}{2} - \widetilde{\varepsilon},$$

hence infinitely many resonances in $\{\Re(s) \geq \frac{\delta(1-2\delta)}{2} - \epsilon\}, \forall \epsilon > 0$.

Concluding the proof: $\delta \in (1/2, 1)$**:**
Use Lemma 8:

$$B\frac{e^{(2\delta-1/2)T}}{T^2} = O\left(\sqrt{\sigma}e^{2\delta T}\right) + O\left(e^{2(\rho+\widetilde{\varepsilon})T}\sigma^{-\delta}\right),$$

which if $\sigma = e^{-\alpha T}$ produces a contradiction whenever $\alpha > 1$ and

$$\rho < \frac{\delta(2-\alpha)}{2} - \frac{1}{4} - \widetilde{\varepsilon}.$$

Hence, infinitely many resonances in the strip

$$\{\Re(s) \geq \frac{\delta}{2} - \frac{1}{4} - \epsilon\},\ \forall \epsilon > 0.$$

Questions:

Lower bound for $\delta = 1/2$?

Lower bounds for "non-arithmetic" groups if $\delta > 1/2$?

Effective upper bounds for the essential spectral gap?