Resonances

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Joint work with F. Naud (Avignon)

- [JN1] *Lower bounds for resonances of infinite area Riemann surfaces.*

- [JN2] *On the resonances of convex co-compact subgroups of arithmetic groups*
  arXiv:1011.6264

1st December 2010
Let $\Gamma$ be a *geometrically finite* subgroup of $\text{PSL}(2, \mathbb{R})$ without elliptic elements. Fundamental domain $X = \Gamma \backslash \mathbb{H}^2$ has finitely many sides. Assume that $X$ has *infinite* hyperbolic area: $X$ decomposes into a finite area surface $N$ (called *Nielsen region* or *convex core*) to which finitely many infinite area half-cylinders (*funnels*) are glued.

If $\Gamma$ has parabolic elements, then $N$ has *cusps* (parabolic vertices); a surface without cusps is called *convex co-compact*; then $\Gamma$ has no parabolic elements.
• The spectrum of \( \Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) \) on \( X \) consists of the continuous spectrum \([1/4, +\infty]\) (no embedded eigenvalues).

• \( \delta \) is the Hausdorff dimension of the limit set \( \Lambda(\Gamma) \subset S^1 \).
  If \( \delta > 1/2 \), \( \Delta \) has finitely many eigenvalues in \((0, 1/4)\); the first nonzero eigenvalue \( \lambda_0 = \delta(1 - \delta) \). Point spectrum is empty if \( \delta \leq 1/2 \) (Lax, Phillips, Patterson, Sullivan).

• The resolvent

\[
R(\lambda) = \left( \Delta - \frac{1}{4} - \lambda^2 \right)^{-1} : L^2(X) \rightarrow L^2(X)
\]

is well-defined and analytic in \( \{\Re(\lambda) < 0\} \), except for finitely many poles corresponding to the finite point spectrum.
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Resonances are the poles of meromorphic continuation of the resolvent \( R(\lambda) : C_0^\infty(X) \to C^\infty(X) \) to the whole complex plane \( \mathbb{C} \). Their set is denoted by \( \mathcal{R}_X \). Guillopé and Zworski showed that \( \exists C > 0 \) such that

\[
\frac{1}{C} < \# \{ z \in \mathcal{R}_X : |z| < R \} / R^2 < C, \quad R \to \infty.
\]

Finer asymptotics: let

\[
N_C(T) = \# \{ z \in \mathcal{R}_X : \Im(z) \leq C, |\Re(z)| \leq T \}.
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Zworski, Guillopé and Lin: “fractal” upper bound

**Theorem 1.** For convex co-compact \( X \),

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N_C(T) = O(T^{1+\delta}); \text{ where } C \text{ is fixed, and } T \to \infty.
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• **Lower bounds:** Guillopé, Zworski: \( \forall \epsilon > 0, \exists C_\epsilon > 0, \) such that

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N_{C_\epsilon}(T) = \Omega(T^{1-\epsilon}).
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The proof uses a wave trace formula for resonances on \( X \) and takes into account contributions from a *single* closed geodesic on \( X \).

• **Question:** Can one improve lower bounds taking into account contributions from *many* closed geodesics on \( X \)?

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Guillopé, Lin, Zworski: let

$$D(z) = \{ \lambda \in \mathcal{R}_X : |\lambda - z| \leq 1 \}$$

Then for all $z : \Im(z) \leq C$, we have $D(z) = O(|\Re(z)|^\delta)$.

Let $A > 0$, and let $W_A$ denote the logarithmic neighborhood of the real axis:

$$W_A = \{ \lambda \in \mathbb{C} : \Im \lambda \leq A \log(1 + |\Re \lambda|) \}$$

Theorem 2. Let $X$ be a geometrically finite hyperbolic surface of infinite area, and let $\delta > 1/2$. Then there exists a sequence $\{z_i\} \in W_A, \Re(z_i) \to \infty$ such that

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• **Corollary:** If $\delta > 1/2$, then $\mathcal{W}_A \cap \mathcal{R}_X$ is different from a lattice.

• Examples of $\Gamma$ such that $\delta(\Gamma) > 1/2$ are easy to construct. Pignataro, Sullivan: fix the topology of $X$. Denote by $l(X)$ the maximum length of the closed geodesics that form the boundary of $N$. Then $\lambda_0(X) \leq C(X)/l(X)$, where $C = C(X)$ depends only on the topology of $X$. By Patterson-Sullivan, $\lambda_0 < 1/4 \iff \delta > 1/2$, so letting $l(X) \to 0$ gives many examples.

• Proof of Theorem 2 uses (a version of Selberg) trace formula due to Guillopé and Zworski, and Dirichlet box principle.
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• Theorem 2 gives a logarithmic lower bound
\[ D(z_i) \geq (\log |\Re(z_i)|)^{\frac{\delta-1/2}{\delta}} - \epsilon \] for an infinite sequence of disks \( D(z_i, 1) \). Conjecture of Guillopè and Zworski would imply that for all \( \epsilon > 0 \) there exists \( \{z_i\} \) such that
\[ D(z_i) \geq |\Re(z_i)|^{\delta-\epsilon}. \]

• Question: can one get polynomial lower bounds for some particular groups \( \Gamma \)?
Answer: Yes. Idea: look at infinite index subgroups of arithmetic groups, and use methods of Selberg-Hejhal.

• Theorem 3. Let \( \Gamma \) be an infinite index geom. finite subgroup of an arithmetic group \( \Gamma_0 \) derived from a quaternion algebra. Let \( \delta(\Gamma) > 3/4 \). Then
\[ \forall \epsilon > 0, \forall A > 0, \] there exists \( \{Z_i\} \subset W_A, \Re(Z_i) \to \infty \), such that
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Key ideas:

- **Number of closed geodesics on $X$:**

  \[
  \#\{\gamma : l(\gamma) < T\} \sim \frac{e^{\delta T}}{\delta T}, \quad T \to \infty.
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- **Number of distinct closed geodesics in the arithmetic case:** for $\Gamma$ derived from a quaternion algebra, one has

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  \#\{L < T : L = l(\gamma)\} \ll e^{T/2}.
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  Accordingly, for $\delta > 1/2$, there exists exponentially large multiplicities in the length spectrum.

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**Ex:** $M_1, M_2 \in \text{SL}(2, \mathbb{Z})$, $\text{tr}M_1 \neq \text{tr}M_2$ then

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Trace formula (Guillopé, Zworski): Let $\psi \in C_0^\infty((0, +\infty))$, and $N$ - Nielsen region. Then (in case there are no cusps)

$$\sum_{\lambda \in \mathcal{R}_X} \hat{\psi}(\lambda) = -\frac{V(N)}{4\pi} \int_0^{+\infty} \frac{\cosh(t/2)}{\sin^2(t/2)} \psi(t) dt$$

$$+ \sum_{\gamma \in \mathcal{P}} \sum_{k \geq 1} \frac{l(\gamma) \psi(kl(\gamma))}{2 \sinh(kl(\gamma)/2)},$$

where $\mathcal{P} = \{\text{primitive closed geodesics on } X\}$. For $\alpha, t \gg 0$, we take

$$\psi_{t,\alpha}(x) = e^{-itx} \psi_0(x - \alpha),$$

where $\psi_0 \in C_0^\infty([-1, 1])$, $\psi \geq 0$, and $\psi_0 = 1$ on $[-1/2, 1/2]$. 

Proof:

Arithmetic case

Strips with infinitely many resonances

Lattice points

Proof of Theorem 5
- Geometric side (sum over closed geodesics):

\[ S_{\alpha,t} = \sum_{\alpha-1 \leq kl(\gamma) \leq \alpha+1} \frac{l(\gamma)\psi_0(kl(\gamma) - \alpha)}{2 \sinh(kl(\gamma)/2)} e^{-itkl(\gamma)}. \]

- Lemma 4: \( \exists A > 0 \) s.t. \( \forall T > 0 \), if we let \( \alpha = 2 \log T - A \), and

\[
J(T) = \int_{T}^{3T} \left(1 - \frac{|t - 2T|}{T}\right) |S_{\alpha,t}|^2 dt,
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then

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Lemma 4 $\Rightarrow$ Theorem 3: Assume for contradiction that for all $z \in W_A$, $\Re(z) \geq R_0$ we have $\mathcal{D}(z) \leq |\Re(z)|^\beta$. Let $\alpha = 2 \log T - A$. We have
\[
\frac{C_2 T^{1+4\delta-3}}{(\log T)^2} \leq J(T) \leq \int_T^{3T} |S_{\alpha,T}|^2 dt.
\]
Assumption implies that
\[
S_{\alpha,T} = O(1 + t^\beta + T^{2\delta-3}).
\]
Integrating, we find that
\[
J(T) = O(T^{2\beta+1}).
\]
This leads to a contradiction if $2\beta + 1 < 1 + 4\delta - 3$, or $\beta < 2\delta - 3/2$, proving Theorem 3.
Proof of Lemma 4 uses the fact that geodesic lengths on $X$ have exponentially high multiplicities and their lengths are well-separated.

After expanding $|S_{\alpha,T}^2|$ and integrating, we write $J(T) = J_1(T) + J_2(T)$, where $J_1(T)$ is the diagonal term

$$J_1(T) = T \sum_{l \in \mathcal{L}(\Gamma)} \frac{(l\# \mu(l))^2 \psi_0^2(l - \alpha)}{4 \sinh^2(l/2)},$$

where $\mathcal{L}_\Gamma$ denotes set of distinct lengths of closed geodesics on $X$; $\mu(l)$ is the multiplicity of $l$; $l\#$ the primitive length of a closed geodesic.

$J_1(T) \geq 0$, and $J_2(T)$ denotes the off-diagonal term. $J_2(T)$ involves integrals $\int_T^{3T} (1 - |t - 2T|/T) e^{i(l_1 - l_2)t} dt$, where $l_1 \leq l_2$. Since distinct $l_j$-s are well-separated, we get cancellation in $J_2(T)$. One can show that $|J_2(T)| \leq J_1(T)/2$ with $\alpha$, $T$ chosen as in Lemma.
• It remains to bound $J_1(T)$ from below. $\psi_0(l - \alpha)$ is supported on $[\alpha - 1, \alpha + 1]$. The denominator $4 \sinh^2(l/2)$ is of order $e^\alpha$. We find that

$$J_1(T) \geq C_3 Te^{-\alpha} \sum_{l \in \mathcal{L} \cap [\alpha - 1/2, \alpha + 1/2]} (\mu(l))^2.$$  

• Call the last sum $S$. Then

$$S \geq \left( \frac{\sum_{l \in \mathcal{L} \cap [\alpha - 1/2, \alpha + 1/2]} \mu(l)}{\left( \sum_{l \in \mathcal{L} \cap [\alpha - 1/2, \alpha + 1/2]} 1 \right)^2} \right)^2$$

The numerator is $\gg [e^{\delta \alpha} / \alpha]^2$ by the prime geodesic theorem. The denominator is $O(e^{\alpha/2})$ (since the lengths are well-separated). Hence $S \gg e^{(2\delta - 1/2)\alpha} / \alpha^2$. Substituting $J(T) \gg S \cdot T / e^\alpha$, $\alpha = 2 \log T - A$, we get $J(T) \gg T^{4\delta - 2} / (\log T)^2$, proving Lemma 4.
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It follows from a recent result of Lewis Bowen that in every co-finite or co-compact arithmetic Fuchsian group, one can find infinite index convex co-compact subgroups with $\delta$ arbitrarily close to 1 (and in particular $> 3/4$). A. Gamburd considered infinite index subgroups of $SL_2(\mathbb{Z})$ and constructed subgroups $\Lambda_N$ such that $\delta(\Lambda_N) \to 1$ as $N \to \infty$. It was shown in [JN1] that subgroups $\Gamma_N$ of $\Lambda_N$ (of index two) provide examples of “arithmetic” groups with $\delta(\Gamma_N) > 3/4$ for large enough $N$. Related questions were also considered by Bourgain and Kantorovich. The results of [JN1] can also be generalized to hyperbolic 3-manifolds (work in progress).
We describe some results in [JN2]. Let \( \lambda = s(1 - s), \, s \in \mathbb{C} \).

- If \( X \) has finite area, then all resonances lie in the strip \( 0 < \Re(s) < 1/2 \).
- If \( X \) has infinite area, then resonances are spread all over the half plane \( \{ \Re(s) < 1/2 \} \). We study resonances with the largest real part.
- If \( \delta > 1/2 \), then all but finitely many resonances lie in the plane \( \{ \Re(s) < 1/2 \} \). If \( \delta \leq 1/2 \), F. Naud showed that there exists \( \epsilon > 0 \) such that
  \[
  \mathcal{R}_X \cap \{ \Re(s) \geq \delta - \epsilon \} = \{ \delta \} .
  \]
  Constant \( \epsilon \) is not effective (follows from a Dolgopyat type estimate).
- We want to find “essential spectral gap”
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  G(\Gamma) := \inf \{ \sigma < \delta : \{ \Re(s) \geq \sigma \} \cap \mathcal{R}_X \text{ is finite} \} .
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- If $\delta > 1/2$, then all but finitely many resonances lie in the plane $\{\Re(s) < 1/2\}$. If $\delta \leq 1/2$, F. Naud showed that there exists $\epsilon > 0$ such that

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- If $X$ has infinite area, then resonances are spread all over the half plane $\{\Re(s) < 1/2\}$. We study resonances with the largest real part.
- If $\delta > 1/2$, then all but finitely many resonances lie in the plane $\{\Re(s) < 1/2\}$. If $\delta \leq 1/2$, F. Naud showed that there exists $\epsilon > 0$ such that

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Constant $\epsilon$ is not effective (follows from a Dolgopyat type estimate).

- We want to find “essential spectral gap”

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  - If $\delta > 1/2$ and $\Gamma$ is a convex co-compact subgroup of an arithmetic group, then $G(\Gamma) \geq \frac{\delta}{2} - \frac{1}{4}$.
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We next discuss connections to lattice point counting problems.

- **Lax-Phillips:** Let \( \delta > 1/2 \). Given \( z, z' \in \mathbb{H} \), let
  \[
  N(T; z, z') := \#\{\gamma \in \Gamma : d(z, \gamma z') \leq T\}.
  \]
  Then
  \[
  N(T; z, z') = \sum_j C_j(z, z') e^{\delta_j T} + O\left(T^{5/6} e^{(\delta+1)T/3}\right),
  \]
  where \( \delta_j \in (1/2, \delta] \), \( \delta_j(1 - \delta_j) = \lambda_j \in [0, 1/4], \delta_0 = \delta \).

- **Conjecture:** optimal error term should be \( O(e^{(\delta/2+\epsilon)T}) \); expansion may contain additional terms.
We next discuss connections to lattice point counting problems.

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- Conjecture: optimal error term should be $O(e^{(\delta/2+\epsilon)T})$; expansion may contain additional terms.
**Theorem 6:** $\Gamma$ convex co-compact subgroup of an arithmetic group with $\delta > 1/2$. There exists a full measure subset $\mathcal{G} \subset \mathbb{H} \times \mathbb{H}$ such that for all $(z, z') \in \mathcal{G}$ and all finite expansion of the form $\sum_j Q_j(T; z, z') e^{\delta_j T}$, where $\delta_j \in \mathbb{C}$ and $Q_j(T; z, z') \in \mathbb{C}[T]$, then for all $\epsilon > 0$

\[
\left| N(T; z, z') - \sum_j Q_j(T; z, z') e^{\delta_j T} \right| = \Omega \left( e^{(\delta/2-1/4-\epsilon)T} \right).
\]

Here $\Omega(\bullet)$ means not a $O(\bullet)$. 

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**General Results**

**Lower bounds**

**Proof:**

Arithmetic case

Strips with infinitely many resonances

Lattice points

**Proof of Theorem 5**
Approximate trace formula: let $\varphi \in C_0^\infty(\mathbb{R})$. Let

$$
\psi(s) := \int_{-\infty}^{+\infty} e^{su} \varphi(u) du = \hat{\varphi}(is),
$$

where $\hat{\varphi}$ F.T. of $\varphi$.

**Proposition 7.** Let $\rho < \delta$, and assume

$$
\# \mathcal{F}_\rho := \# \mathcal{R}_X \cap \{\Re(s) > \rho\} < \infty.
$$

Then $\forall \varepsilon > 0$ small enough, $\exists \varepsilon \leq \bar{\varepsilon} \leq 2\varepsilon$ s.t.

$$
\sum_{k \in \mathbb{N}_0} \sum_{\gamma \in \mathcal{P}} \frac{l(\gamma)}{1 - e^{-kl(\gamma)}} \varphi(kl(\gamma)) = \sum_{\lambda \in \mathcal{F}_\rho} \psi(\lambda)
$$

$$
+ O \left( \int_{-\infty}^{+\infty} (1 + |x|)^\delta |\psi(\rho + \bar{\varepsilon} + ix)| dx \right).
$$

Here $\mathcal{P}$ - primitive closed geodesics. Constant depends on $\varepsilon, \rho$ and $\Gamma$. 
Lower bound on multiplicities for arithmetic groups: For all $\ell \in \mathcal{L}_\Gamma$, let

$$m(\ell) := \#\{(k, \gamma) \in \mathbb{N}_0 \times \mathcal{P} : \ell = kl(\gamma)\}.$$ 

**Lemma 8.** Assume $\delta(\Gamma) > 1/2$. Then $\exists A_\Gamma > 0$ such that for all $T$ large, we have

$$\sum_{T-1 \leq \ell \leq T+1} m^2(\ell) \geq A_\Gamma \frac{e^{(2\delta - 1/2)T}}{T^2}.$$ 

Proof uses *bounded clustering* property of $\Gamma$ (Luo, Sarnak).
Proof of Theorem 5. Test functions: Let $\xi \in \mathbb{R}$, $T \gg 0$, and

$$\varphi_{\xi,T}(x) := e^{-i\xi x} \varphi(x - T),$$

where

$$\varphi \in C_0^\infty([-2, 2]); \varphi \geq 0; \varphi(x) = 1, \quad x \in [-1, 1].$$

Let $A \geq \Re(s) \geq 0$, then

$$\psi_{\xi,T}(s) := \widehat{\varphi_{\xi,T}}(is) = e^{-i\xi s} e^{s T} \hat{\varphi}(\xi + is).$$
Let

\[ S_{\xi,T} := \sum_{k \in \mathbb{N}_0} \sum_{\gamma \in \mathcal{P}} \frac{l(\gamma)}{1 - e^{-k l(\gamma)}} e^{-i \xi k l(\gamma)} \varphi(k l(\gamma) - T). \]

Approximate trace formula (Prop. 7) implies

\[ S_{\xi,T} = \sum_{\lambda \in \mathcal{F}_\rho} \psi_{\xi,T}(\lambda) + E(\xi, T), \]

where

\[ E(\xi, T) = O\left(e^{(\rho + \tilde{\varepsilon}) T |\xi| \delta}\right). \]
Consider
\[ G(\sigma, T) := \sqrt{\sigma} \int_{-\infty}^{+\infty} e^{-\sigma \xi^2} |S_{\xi, T}|^2 d\xi, \]
where \( \sigma = \sigma(T) > 0 \) - small. One can show that
\[ G(\sigma, T) = \sqrt{\pi} \sum_{\ell, \ell' \in L} a_{\ell, \ell'} \varphi(\ell - T) \varphi(\ell' - T) e^{-\frac{(\ell - \ell')^2}{4\sigma}}, \]
where
\[ a_{\ell, \ell'} := \frac{\widetilde{\ell \ell'} m(\ell) m(\ell')}{(1 - e^{-\ell})(1 - e^{-\ell'})}. \]
It follows that
\[ G(\sigma, T) \geq C \sum_{T-1 \leq \ell \leq T+1} m^2(\ell). \]
Proposition 7 allows to bound \( \frac{g(\sigma, T)}{2\sqrt{\sigma}} \) by

\[
\int_{-\infty}^{+\infty} e^{-\sigma \xi^2} \left| \sum_{\lambda \in F_{\rho}} \psi_{\xi, T}(\lambda) \right|^2 d\xi + \int_{-\infty}^{+\infty} e^{-\sigma \xi^2} |E(\xi, T)|^2 d\xi.
\]

We assume that \( F_{\rho} \) is finite, hence

\[
I_1(\sigma, T) = O\left(e^{2\delta T}\right),
\]

uniformly in \( \sigma \). Also, one can show

\[
I_2(\sigma, T) = O\left(e^{2(\rho + \bar{\epsilon}) T \sigma^{-\delta - 1/2}}\right).
\]
Concluding the proof: \( \delta \in (0, 1/2) \): Cannot use Lemma 8. Use

\[
G(\sigma, T) \geq C \sum_{T-1 \leq \ell \leq T+1} m(\ell) \geq B \frac{e^{\delta T}}{T},
\]

(using prime geodesic theorem), for \( B > 0 \). Let \( T \gg 0, \sigma \ll 1 \).

\[
B \frac{e^{\delta T}}{T} = O \left( \sqrt{\sigma} e^{2\delta T} \right) + O \left( e^{2(\rho + \tilde{\epsilon}) T} \sigma^{-\delta} \right).
\]

Let \( \sigma = e^{-\alpha T} \); get a contradiction as \( T \to +\infty \) if

\[
\alpha > 2\delta \text{ and } \rho < \frac{\delta(1 - \alpha)}{2} - \tilde{\epsilon},
\]

hence infinitely many resonances in

\[
\{ \Re(s) \geq \frac{\delta(1-2\delta)}{2} - \epsilon \}, \forall \epsilon > 0.
\]
Concluding the proof: $\delta \in (1/2, 1)$:

Use Lemma 8:

$$B\frac{e^{(2\delta - 1/2)T}}{T^2} = O\left(\sqrt{\sigma}e^{2\delta T}\right) + O\left(e^{2(\rho + \tilde{\epsilon})T}\sigma^{-\delta}\right),$$

which if $\sigma = e^{-\alpha T}$ produces a contradiction whenever $\alpha > 1$ and

$$\rho < \frac{\delta(2 - \alpha)}{2} - \frac{1}{4} - \tilde{\epsilon}.$$ 

Hence, infinitely many resonances in the strip

$$\{\Re(s) \geq \frac{\delta}{2} - \frac{1}{4} - \epsilon\}, \quad \forall \epsilon > 0.$$
Questions:
Lower bound for $\delta = 1/2$?
Lower bounds for “non-arithmetic” groups if $\delta > 1/2$?
Effective upper bounds for the essential spectral gap?