Estimates from below for the spectral function and for the remainder in Weyl's law

D. Jakobson (McGill)
email: jakobson@math.mcgill.ca

I. Polterovich (Université de Montréal)
email: iossif@dms.umontreal.ca

J. Toth (McGill)
email: jtoth@math.mcgill.ca

math.SP/0505400

July 25, 2006

$X^n, n \geq 2$ - compact. Δ - Laplacian. Spectrum: $\Delta\phi_i + \lambda_i\phi_i = 0, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$

Eigenvalue counting function:
$N(\lambda) = \#\{\sqrt{\lambda_j} \leq \lambda\}$.

Weyl's law: $N(\lambda) = C_n V \lambda^n + R(\lambda),\ R(\lambda) = O(\lambda^{n-1})$. $R(\lambda)$ - **remainder**.

Spectral function: Let $x, y \in X$.
$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y)$.

If $x = y$, let $N_{x,y}(\lambda) := N_x(\lambda)$.

Local Weyl's law:
$N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y;$
$N_x(\lambda) = C_n\lambda^n + R_x(\lambda), \qquad R_x(\lambda) = O(\lambda^{n-1});$

$R_x(\lambda)$ - **local remainder**. We study **lower** bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda)$.

Notation: $f_1(\lambda) = \Omega(f_2(\lambda)), f_2 > 0$ iff $\limsup_{\lambda\to\infty} |f_1(\lambda)|/f_2(\lambda) > 0$.

Theorem 1[JP] If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

Theorem 2[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

Other results in dimension $n > 2$ involve heat kernel asymptotics.

Example: flat square 2-torus
$\lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbf{Z}$
$\phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}, \; x = (x_1, x_2)$

$$|\phi_j(x)| = 1 \; \Rightarrow \; N(\lambda) \equiv N_x(\lambda)$$

Gauss circle problem: estimate $R(\lambda)$.
Theorem 2 $\Rightarrow$ $R(\lambda) = \Omega(\sqrt{\lambda})$ - **Hardy–Landau bound**. Theorem 2 generalizes that bound for the *local* remainder.

Hardy's conjecture: $R(\lambda) \ll \lambda^{1/2+\epsilon} \; \forall \epsilon > 0$.

Huxley: $R(\lambda) \; \ll \; \lambda^{\frac{46}{73}} \, (\log \lambda)^{\frac{315}{146}}$.

Negative curvature. Suppose sectional curvature satisfies
$-K_1^2 \leq K(\xi, \eta) \leq -K_2^2$

Theorem (Berard): $R_x(\lambda) = O(\lambda^{n-1}/\log \lambda)$

Conjecture (Randol): On a negatively-curved surface, $R(\lambda) = O(\lambda^{\frac{1}{2}+\epsilon})$. Randol proved an integrated (in λ) version for $N_{x,y}(\lambda)$.
Conjecture (?) On a *generic* negatively curved surface

$$R(\lambda) = O(\lambda^{\epsilon}) \qquad \forall \epsilon > 0.$$

Theorem (Karnaukh) On a negatively curved surface

$$R_x(\lambda) = \Omega(\sqrt{\lambda})$$

+ logarithmic improvements discussed below. Karnaukh's results (unpublished 1996 Princeton Ph.D. thesis under the supervision of P. Sarnak) served as a starting point and a motivation for our work.

Thermodynamic formalism: G^t - geodesic flow on SX. $\xi \in SX$, $U(\xi)$ - unstable subspace of $T_\xi SX$ for for G^t.
Sinai-Ruelle-Bowen potential $\mathcal{H} : SM \to \mathbf{R}$:

$$\mathcal{H}(\xi) = \frac{d}{dt}\Big|_{t=0} \ln \det dG^t|_{U(\xi)}$$

Topological pressure $P(f)$ of a Hölder function $f : SX \to \mathbf{R}$ satisfies (Parry, Pollicott)

$$\sum_{l(\gamma)\le T} l(\gamma) \exp\left[\int_\gamma f(\gamma(s), \gamma'(s))ds\right] \sim \frac{e^{P(f)T}}{P(f)}.$$

γ - geodesic of length $l(\gamma)$. $P(f)$ is defined as

$$P(f) = \sup_\mu \left(h_\mu + \int f d\mu\right),$$

μ is G^t-invariant, h_μ - (measure-theoretic) entropy.

Ex. 1: $P(0) = h$ - **topological entropy** of G^t.
Ex. 2: $P(-\mathcal{H}) = 0$. The *equilibrium measure* (attaining the supremum) for $\mathcal{H}$ is the Liouville measure μ_L on SX, thus $h_{\mu_L} = \int_{SX} \mathcal{H} d\mu_L$.

Theorem 3[JP] If X is negatively-curved then for any $\delta > 0$ and $x \neq y$

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

Here $P(-\mathcal{H}/2)/h \geq K_2/(2K_1) > 0$.

Theorem 4a[JP] X - negatively-curved. For any $\delta > 0$

$$R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right), \ n = 2, 3.$$

Results for $n \geq 4$ involve heat kernel asymptotics.

$$K = -1 \ \Rightarrow R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{1}{2} - \delta}\right)$$

Karnaukh, $n = 2$: estimate above + weaker estimates in variable negative curvature.

Global results: $R(\lambda)$

Randol:

$$K = -1 \ \Rightarrow R(\lambda) = \Omega\left((\log \lambda)^{\frac{1}{2}-\delta}\right), \qquad \forall \delta > 0.$$

Theorem 4b[JPT] X - negatively-curved. For any $\delta > 0$

$$R(\lambda) = \Omega\left((\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right), \ n = 2.$$

Results for $n \geq 3$ involve heat trace asymptotics.

Selberg, Hejhal: On compact arithmetic surfaces that correspond to quaternionic lattices $R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}}{\log \lambda}\right)$. **Reason:** high multiplicities in the length spectrum; generically, X has *simple* length spectrum.

Proof of Theorem 4b: $R(\lambda)$.

Wave trace on X:

$$e(t) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t).$$

$\psi \in C_0^\infty([-1,1])$, even, monotone decreasing on [0,1], $\psi \geq 0$, $\psi(0) = 1$. Fix $\lambda, T \gg 0$, consider the function

$$(1/T)\psi(t/T)\cos(\lambda t).$$

We let

$$k_{\lambda,T} = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T}\cos(\lambda t)e(t)dt$$

Lemma 5. If $R(\lambda) = o((\log\lambda)^b)$, $b > 0$ then

$$k_{\lambda,T} = o((\log\lambda)^b).$$

Goal: estimate $k_{\lambda,T}$ from below. Need to extract long exponential sums as the leading asymptotics of the long-time wave trace expansion.

Key microlocal result:

Proposition 6. Let $\lambda, T \to \infty,\ T \leq \epsilon \log \lambda$ for small enough $\epsilon > 0$. Then $\exists\, \delta(\epsilon) > 0$ such that $\kappa_{\lambda,T} = \sum_{l(\gamma) \leq T} \frac{l(\gamma)^{\#} \cos(\lambda l(\gamma))\, \psi\left(\frac{l(\gamma)}{T}\right)}{2T\sqrt{|\det(I - \mathcal{P}_\gamma)|}} + \mathcal{O}\left(\lambda^{-\delta(\epsilon)}\right)$,
where $\mathcal{P}_\gamma$ - Poincaré map.

Quantitative version of the "wave trace" formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed geodesics of size $\sim \lambda^{-\tau}, \tau$ small. Allows to isolate contribution from a **growing number** of closed geodesics with $l(\gamma) \leq T$ to $\kappa(\lambda, T)$ as $\lambda, T \to \infty$.

Proof - separation of closed geodesics in phase space + small-scale microlocalization near closed geodesics.

Consider the sum

$$S(T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}}$$

$\mathcal{P}_\gamma$ preserves stable and unstable subspaces. Dimension 2: eigenvalues are $\exp\left[\pm \int_\gamma \mathcal{H}(\gamma(s), \gamma'(s)) ds\right]$.

$\mathcal{P}_\gamma - Id$ is conjugate to

$$\begin{pmatrix} \exp\left[\int_\gamma \mathcal{H}\right] - 1 & 0 \\ 0 & \exp\left[-\int_\gamma \mathcal{H}\right] - 1 \end{pmatrix}$$

Thus, $S(T)$ is asymptotic to

$$\sum_{l(\gamma) \leq T} l(\gamma) \exp\left[-\frac{1}{2}\int_\gamma \mathcal{H}\right].$$

Results of Parry and Pollicott $\Rightarrow$

Theorem 7. As $T \to \infty$,

$$S(T) \sim \frac{e^{P\left(-\frac{\mathcal{H}}{2}\right) \cdot T}}{P(-\mathcal{H}/2)}$$

Here $P\left(-\frac{\mathcal{H}}{2}\right) \geq (n-1)K_2/2$.

Dirichlet box principle $\Rightarrow$ "straighten the phases:" $\exists\lambda$ s.t.

$$\cos(\lambda l(\gamma)) > \nu > 0, \ \forall\gamma : l(\gamma) \leq T.$$

($\lambda l(\gamma)$ close to $2\pi\mathbf{Z}$). This combined with Theorem 7 shows that $\exists\lambda, T$ s.t.

$$\kappa_{\lambda,T} \sim \frac{\exp[P\left(-\frac{\mathcal{H}}{2}\right)T(1-\delta/2)]}{T}$$

This leads to contradiction with Lemma 5. Q.E.D.

For Dirichlet principle need

$$T \asymp \ln\ln\lambda.$$

So, get logarithmic lower bound in Theorem 4b.

Proof of Theorem 3: $N(x,y,\lambda)$

Wave kernel on X:

$$e(t,x,y) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t)\phi_i(x)\phi_i(y),$$

fundamental solution of the wave equation $(\partial^2/\partial t^2 - \Delta)e(t,x,y) = 0,\ e(0,x,y) = \delta(x-y),$ $(\partial/\partial t)e(0,x,y) = 0.$

$$k_{\lambda,T}(x,y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t)e(t,x,y)dt$$

Lemma 5a If $N_{x,y}(\lambda) = o(\lambda^a(\log\lambda)^b))$, where $a > 0, b > 0$ then

$$k_{\lambda,T}(x,y) = o(\lambda^a(\log\lambda)^b)).$$

Pretrace formula. M - universal cover of X, no conjugate points, $E(t,x,y)$ be the wave kernel on M. Then for $x, y \in X$, we have

$$e(t,x,y) = \sum_{\omega\in\pi_1(X)} E(t,x,\omega y)$$

Hadamard Parametrix for $E(t,x,y) \Rightarrow$

$$K_{\lambda,T}(x,y) \sim_{\lambda\to\infty} Q_1 \lambda^{\frac{n-1}{2}} \times \sum_{\omega\in\pi_1(X):d(x,\omega y)\le T}$$

$$\frac{\psi\left(\frac{d(x,\omega y)}{T}\right)\sin(\lambda d(x,\omega y)+\theta_n)}{\sqrt{Tg(x,\omega y)\,d(x,\omega y)^{n-1}}} + O\left[\lambda^{\frac{n-3}{2}} e^{O(T)}\right].$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3 - (n \bmod 8))$, and $Q_1 \neq 0$.

Pointwise analog of the sum $S(T)$:

$$S_{x,y}(T) = \sum_{\omega:d(x,\omega y)\le T} \frac{1}{\sqrt{g(x,\omega y)\,d(x,\omega y)^{n-1}}},$$

where $g = \sqrt{\det g_{ij}}$ in normal coordinates at x. $S_{x,y}(T)$ grows at the same rate as $S(T)$.

Reason: let $x, y \in M$, γ - geodesic from x to y, $\xi = (x, \gamma'(0))$, and dist$(x, y) = r$. Then $\sqrt{g(x,y)r^{n-1}} \ll Jac_{Vert(\xi)} G^r$.

Here $Vert(\xi) \in T_\xi SM$ - vertical subspace; $U(\xi) \in T_\xi SM$ - unstable subspace at ξ.

By properties of Anosov flows,

$$\text{Dist}[DG^r(Vert(\xi)), DG^r(U(\xi))] \leq Ce^{-\alpha r}.$$

Therefore,

$$Jac_{Vert(\xi)} G^r \ll Jac_{U(\xi)} G^r = \exp\left[\int_\gamma \mathcal{H}\right]$$

Our **local** estimates are not uniform in x, y. Need Proposition 6 to prove **global** estimates.

Heat trace asymptotics:

$$\sum_i e^{-\lambda_i t} \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j \; t^{j-\frac{n}{2}}, \qquad t \to 0^+$$

Local: $\mathcal{K}(t,x,x) = \sum_i e^{-\lambda_i t}\phi_i^2(x) \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j-\frac{n}{2}}$,

$a_j(x)$ - local heat invariants, $a_j = \int_X a_j(x)dx$.

$a_0(x) = 1$, $a_0 = \mathrm{vol}(X)$. $a_1(x) = \frac{\tau(x)}{6}$, $\tau(x)$ - scalar curvature.

"Heat kernel" estimates:

Theorem 2b[JP] If the scalar curvature $\tau(x) \neq 0, \Longrightarrow R_x(\lambda) = \Omega(\lambda^{n-2})$.
Global:[JPT] If $\int_X \tau \neq 0, \Rightarrow R(\lambda) = \Omega(\lambda^{n-2})$.

Remark: if $\tau(x) = 0$, let $k = k(x)$ be the first positive number such that the k-th local heat invariant $a_k(x) \neq 0$. If $n - 2k(x) > 0$, then

$$R_x(\lambda) = \Omega(\lambda^{n-2k(x)}).$$

Similar result holds for $R(\lambda)$: if $\int a_k(x)dx \neq 0$ and $n - 2k > 0$, then

$$R(\lambda) = \Omega(\lambda^{n-2k}).$$

Oscillatory error term: subtract $[(n-1)/2]$ terms coming from the heat trace:

$$N_x(\lambda) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \frac{a_j(x)\lambda^{n-2j}}{(4\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}-j+1)} + R_x^{osc}(\lambda)$$

Warning: **not** an asymptotic expansion!

Physicists: subtract the "mean smooth part" of $N_x(\lambda)$.

Theorem 2c[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x^{osc}(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

Theorem 4c[JP] X - negatively-curved. For any $\delta > 0$

$$R_x^{osc}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right), \ any \ n.$$

If $n \geq 4$ then Theorem 2b, $R_x(\lambda) = \Omega(\lambda^{n-2})$ gives a better bound for $R_x(\lambda)$.

Global results:

Theorem 4d[JPT] X - negatively-curved. For any $\delta > 0$
$R^{osc}(\lambda) = \Omega\left((\log \lambda)^{\frac{P(-\mathcal{H}/2)}{\hbar}-\delta}\right),\ any\ n.$

If $n \geq 3$ then Theorem 2b, $R(\lambda) = \Omega(\lambda^{n-2})$ gives a better bound for $R(\lambda)$.

Concluding remarks

• On a negatively-curved surface, $R_x(\lambda) = \Omega(\sqrt{\lambda})$, while $R(\lambda)$ is conjectured to be $O(\lambda^\epsilon)$.

• **Papers:**

"Estimates from below for the spectral function and for the remainder in local Weyl's law." [JP], math.SP/0505400, submitted. Announced in ERA-AMS **11** (2005), 71–77.

"Lower bounds for the remainder in Weyl's law on negatively curved manifolds." [JPT], in preparation.

Local Weyl's law (IHP, July 2005 talk):
www.math.mcgill.ca/jakobson/papers/ihp05.pdf