

# Estimates from below for the spectral function and for the remainder in Weyl's law

D. Jakobson (McGill)

email: jakobson@math.mcgill.ca

I. Polterovich (Université de Montréal)

email: iossif@dms.umontreal.ca

J. Toth (McGill)

email: jtoth@math.mcgill.ca

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$X^n, n \geq 2$  - compact.  $\Delta$  - Laplacian. Spectrum:  $\Delta\phi_i + \lambda_i\phi_i = 0, 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$

**Eigenvalue counting function:**

$$N(\lambda) = \#\{\sqrt{\lambda_j} \leq \lambda\}.$$

**Weyl's law:**  $N(\lambda) = C_n V \lambda^n + R(\lambda), R(\lambda) = O(\lambda^{n-1}). R(\lambda)$  - remainder.

**Spectral function:** Let  $x, y \in X$ .

$$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y).$$

If  $x = y$ , let  $N_{x,y}(\lambda) := N_x(\lambda)$ .

**Local Weyl's law:**

$$N_{x,y}(\lambda) = O(\lambda^{n-1}), \quad x \neq y;$$
$$N_x(\lambda) = C_n \lambda^n + R_x(\lambda), \quad R_x(\lambda) = O(\lambda^{n-1});$$

$R_x(\lambda)$  - local remainder. We study lower bounds for  $R(\lambda), R_x(\lambda)$  and  $N_{x,y}(\lambda)$ .

Notation:  $f_1(\lambda) = \Omega(f_2(\lambda))$ ,  $f_2 > 0$  iff  
 $\limsup_{\lambda \rightarrow \infty} |f_1(\lambda)|/f_2(\lambda) > 0$ .

**Theorem 1**[JP] If  $x, y \in X$  are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

**Theorem 2**[JP] If  $x \in X$  is not conjugate to itself along any shortest geodesic loop, then

$$R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right)$$

Other results in dimension  $n > 2$  involve heat invariants.

**Example: flat square 2-torus**

$$\lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbf{Z}$$

$$\phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}, \quad x = (x_1, x_2)$$

$$|\phi_j(x)| = 1 \Rightarrow N(\lambda) \equiv N_x(\lambda)$$

**Gauss circle problem:** estimate  $R(\lambda)$ .

Theorem 2  $\Rightarrow$   $R(\lambda) = \Omega(\sqrt{\lambda}) -$

**Hardy–Landau bound.** Theorem 2 generalizes that bound for the *local* remainder.

**Hardy's conjecture:**  $R(\lambda) \ll \lambda^{1/2+\epsilon} \forall \epsilon > 0$ .

**Huxley:**  $R(\lambda) \ll \lambda^{\frac{131}{208}} (\log \lambda)^{2.26}$ .

**Negative curvature.** Suppose sectional curvature satisfies

$$-K_1^2 \leq K(\xi, \eta) \leq -K_2^2$$

**Theorem (Berard):**  $R_x(\lambda) = O(\lambda^{n-1} / \log \lambda)$

**Conjecture (Randol):** On a negatively-curved surface,  $R(\lambda) = O(\lambda^{\frac{1}{2}+\epsilon})$ . Randol proved an integrated (in  $\lambda$ ) version for  $N_{x,y}(\lambda)$ .

**Theorem (Karnaukh)** On a negatively curved surface

$$R_x(\lambda) = \Omega(\sqrt{\lambda})$$

+ logarithmic improvements discussed below. Karnaukh's results (unpublished 1996 Princeton Ph.D. thesis under the supervision of P. Sarnak) served as a starting point and a motivation for our work.

**Thermodynamic formalism:**  $G^t$  - geodesic flow on  $SX$ ,  $\xi \in SX$ ,  $T_\xi(SX) = E_\xi^s \oplus E_\xi^u \oplus E_\xi^o$ ,

- $\dim E_\xi^s = n - 1$  : stable subspace, exponentially contracting for  $G^t$ ;
- $\dim E_\xi^u = n - 1$  : unstable subspace, exponentially contracting for  $G^{-t}$ ;
- $\dim E_\xi^o = 1$  : tangent subspace to  $G^t$ .

**Sinai-Ruelle-Bowen potential  $\mathcal{H} : SM \rightarrow \mathbf{R}$ :**

$$\mathcal{H}(\xi) = \frac{d}{dt} \Big|_{t=0} \ln \det dG^t|_{E_\xi^u}$$

**Topological pressure**  $P(f)$  of a Hölder function  $f : SX \rightarrow \mathbf{R}$  satisfies (Parry, Pollicott)

$$\sum_{l(\gamma) \leq T} l(\gamma) \exp \left[ \int_{\gamma} f(\gamma(s), \gamma'(s)) ds \right] \sim \frac{e^{P(f)T}}{P(f)}.$$

$\gamma$  - geodesic of length  $l(\gamma)$ .  $P(f)$  is defined as

$$P(f) = \sup_{\mu} \left( h_{\mu} + \int f d\mu \right),$$

$\mu$  is  $G^t$ -invariant,  $h_{\mu}$  - (measure-theoretic) entropy.

Ex 1:  $P(0) = h$  - **topological entropy** of  $G^t$ . Theorem (Margulis):  $\#\{\gamma : l(\gamma) \leq T\} \sim e^{hT} / hT$ .

Ex. 2:  $P(-\mathcal{H}) = 0$ .

**Theorem 3**[JP] If  $X$  is negatively-curved then for any  $\delta > 0$  and  $x \neq y$

$$N_{x,y}(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right)$$

Here  $P(-\mathcal{H}/2)/h \geq K_2/(2K_1) > 0$ .

**Theorem 4a**[JP]  $X$  - negatively-curved. For any  $\delta > 0$

$$R_x(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right), \quad n = 2, 3.$$

Results for  $n \geq 4$  involve heat invariants.

$$K = -1 \Rightarrow R_x(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{1}{2} - \delta} \right)$$

**Karnaukh**,  $n = 2$ : estimate above + weaker estimates in variable negative curvature.

**Global results:**  $R(\lambda)$

**Randol**,  $n = 2$ :

$$K = -1 \Rightarrow R(\lambda) = \Omega \left( (\log \lambda)^{\frac{1}{2} - \delta} \right), \quad \forall \delta > 0.$$

**Theorem 4b**[JPT]  $X$  - negatively-curved surface ( $n = 2$ ). For any  $\delta > 0$

$$R(\lambda) = \Omega \left( (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right).$$

**Conjecture (folklore).** On a **generic** negatively curved surface

$$R(\lambda) = O(\lambda^\epsilon) \quad \forall \epsilon > 0.$$

**Selberg, Hejhal:** On compact arithmetic surfaces that correspond to quaternionic lattices  $R(\lambda) = \Omega \left( \frac{\sqrt{\lambda}}{\log \lambda} \right)$ . **Reason:** *exponentially high* multiplicities in the length spectrum; generically,  $X$  has *simple* length spectrum.

**Proof of Theorem 4b:** (about  $R(\lambda)$ ).  $X$ -compact, negatively-curved surface.

**Wave trace** on  $X$  (even part):

$$e(t) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t).$$



- Cut-off:**  $\chi(t, T) = (1 - \psi(t))\hat{\rho}\left(\frac{t}{T}\right)$ , where
- $\rho \in \mathcal{S}(\mathbf{R})$ ,  $\text{supp } \hat{\rho} \subset [-1, +1]$ ,  $\rho \geq 0$ , even;
  - $\psi(t) \in C_0^\infty(\mathbf{R})$ ,  $\psi(t) \equiv 1, t \in [-T_0, T_0]$ , and  $\psi(t) \equiv 0, |t| \geq 2T_0$ .

In the sequel,  $T = T(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ . Let

$$\kappa(\lambda, T) = \frac{1}{T} \int_{-\infty}^{\infty} e(t)\chi(t, T) \cos(\lambda t) dt$$

**Key microlocal result:**

**Proposition 5.** Let  $T = T(\lambda) \leq \epsilon \log \lambda$ . Then

$$\kappa(\lambda, T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)^{\#} \cos(\lambda l(\gamma)) \cdot \chi(l(\gamma), T)}{T \sqrt{|\det(I - \mathcal{P}_\gamma)|}}$$

$+O(1)$ , where

- $\gamma$  - closed geodesic;  $l(\gamma)$  - length;  $l(\gamma)^{\#}$  - primitive period;  $\mathcal{P}_\gamma$  - Poincaré map.

*Long-time* version of the “wave trace” formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed

geodesics. Allows to isolate contribution from a **growing number** of closed geodesics with  $l(\gamma) \leq T(\lambda)$  to  $\kappa(\lambda, T)$  as  $\lambda, T(\lambda) \rightarrow \infty$ .

**Proof** - separation of closed geodesics in phase space + small-scale microlocalization near closed geodesics.

**Dynamical result** (“spaghetti lemma”): Let  $X$  - compact, negatively curved manifold.  $\Omega(\gamma, r)$  - neighborhood of  $\gamma$  in  $S^*X$  of radius  $r$ . There exist constants  $B > 0, a > 0$  s.t. for all closed geodesics on  $X$  with  $l(\gamma) \in [T-a, T]$ , the neighborhoods  $\Omega(\gamma, e^{-BT})$  are disjoint, provided  $T > T_0$ .

Radius  $r = e^{-BT}$  is exponentially small in  $T$ , since the number of closed geodesic grows exponentially.

**Question:** is Lemma true without assuming that  $l(\gamma) \in [T - a, T]$ , i.e. that the lengths are close to each other?

**Lemma 6.** If  $R(\lambda) = o((\log \lambda)^b)$ ,  $b > 0$  then

$$\kappa(\lambda, T) = o((\log \lambda)^b).$$

**Goal:** estimate  $\kappa(\lambda, T)$  from below. Need to extract long exponential sums as the leading asymptotics of the long-time wave trace expansion.

Consider the sum

$$S(T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}}$$

$\mathcal{P}_\gamma$  preserves stable and unstable subspaces.  
Dimension 2: eigenvalues are  $\exp \left[ \pm \int_\gamma \mathcal{H}(\gamma(s), \gamma'(s)) ds \right]$ .

$\mathcal{P}_\gamma - Id$  is conjugate to

$$\begin{pmatrix} \exp \left[ \int_\gamma \mathcal{H} \right] - 1 & 0 \\ 0 & \exp \left[ - \int_\gamma \mathcal{H} \right] - 1 \end{pmatrix}$$

Thus,  $S(T)$  is asymptotic to

$$\sum_{l(\gamma) \leq T} l(\gamma) \exp \left[ - \frac{1}{2} \int_\gamma \mathcal{H} \right].$$

Results of Parry and Pollicott  $\Rightarrow$

**Theorem 7.** As  $T \rightarrow \infty$ ,

$$S(T) \sim \frac{e^{P\left(-\frac{\mathcal{H}}{2}\right) \cdot T}}{P(-\mathcal{H}/2)}$$

Here  $P\left(-\frac{\mathcal{H}}{2}\right) \geq (n-1)K_2/2$ .

**Dirichlet box principle**  $\Rightarrow$  “straighten the phases:”

$\exists \lambda$  s.t.

$$\cos(\lambda l(\gamma)) > \nu > 0, \quad \forall \gamma : l(\gamma) \leq T.$$

( $\lambda l(\gamma)$  close to  $2\pi\mathbf{Z}$ ). This combined with Theorem 7 shows that  $\exists \lambda, T$  s.t.

$$\kappa(\lambda, T) \sim \frac{\exp\left[P\left(-\frac{\mathcal{H}}{2}\right) T(1 - \delta/2)\right]}{T}$$

This leads to contradiction with Lemma 6.  
Q.E.D.

For Dirichlet principle need  $T \asymp \ln \ln \lambda$ , So, get logarithmic lower bound in Theorem 4b.

### Proof of Theorem 3: $N(x, y, \lambda)$

**Wave kernel** on  $X$ :

$$e(t, x, y) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t) \phi_i(x) \phi_i(y),$$

fundamental solution of the wave equation  
 $(\partial^2/\partial t^2 - \Delta)e(t, x, y) = 0$ ,  $e(0, x, y) = \delta(x - y)$ ,  
 $(\partial/\partial t)e(0, x, y) = 0$ .

$$k_{\lambda, T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) e(t, x, y) dt$$

**Lemma 5a** If  $N_{x,y}(\lambda) = o(\lambda^a(\log \lambda)^b)$ , where  $a > 0, b > 0$  then

$$k_{\lambda, T}(x, y) = o(\lambda^a(\log \lambda)^b).$$

**Pretrace formula.**  $M$  - universal cover of  $X$ , no conjugate points,  $E(t, x, y)$  be the wave kernel on  $M$ . Then for  $x, y \in X$ , we have

$$e(t, x, y) = \sum_{\omega \in \pi_1(X)} E(t, x, \omega y)$$

Hadamard Parametrix for  $E(t, x, y) \Rightarrow$

$$K_{\lambda, T}(x, y) \sim_{\lambda \rightarrow \infty} Q_1 \lambda^{\frac{n-1}{2}} \times \sum_{\omega \in \pi_1(X): d(x, \omega y) \leq T} \frac{\psi\left(\frac{d(x, \omega y)}{T}\right) \sin(\lambda d(x, \omega y) + \theta_n)}{\sqrt{T g(x, \omega y) d(x, \omega y)^{n-1}}} + O\left[\lambda^{\frac{n-3}{2}} e^{O(T)}\right].$$

Here  $g = \sqrt{\det g_{ij}}$  in normal coordinates,  $\theta_n = (\pi/4)(3 - (n \bmod 8))$ , and  $Q_1 \neq 0$ .

Pointwise analog of the sum  $S(T)$ :

$$S_{x,y}(T) = \sum_{\omega: d(x, \omega y) \leq T} \frac{1}{\sqrt{g(x, \omega y) d(x, \omega y)^{n-1}}},$$

where  $g = \sqrt{\det g_{ij}}$  in normal coordinates at  $x$ .  $S_{x,y}(T)$  grows at the same rate as  $S(T)$ .

**Reason:** let  $x, y \in M$ ,  $\gamma$  - geodesic from  $x$  to  $y$ ,  $\xi = (x, \gamma'(0))$ , and  $\text{dist}(x, y) = r$ . Then  $\sqrt{g(x, y) r^{n-1}} \ll \text{Jac}_{\text{Vert}(\xi)} G^r$ .

Here  $Vert(\xi) \in T_\xi SM$  - vertical subspace;  $E_\xi^u \in T_\xi SM$  - unstable subspace at  $\xi$ .

By properties of Anosov flows,

$$\text{Dist}[DG^r(Vert(\xi)), DG^r(E_\xi^u)] \leq Ce^{-\alpha r}.$$

Therefore,

$$Jac_{Vert(\xi)}G^r \ll Jac_{E_\xi^u}G^r = \exp \left[ \int_\gamma \mathcal{H} \right]$$

Our **local** estimates are not uniform in  $x, y$ .  
Need Proposition 5 to prove **global** estimates.

**Heat trace asymptotics:**

$$\sum_i e^{-\lambda_i t} \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j t^{j-\frac{n}{2}}, \quad t \rightarrow 0^+$$

**Local:**  $\mathcal{K}(t, x, x) = \sum_i e^{-\lambda_i t} \phi_i^2(x) \sim$   
 $\frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j-\frac{n}{2}},$



$a_j(x)$  - local heat invariants,  $a_j = \int_X a_j(x) dx$ .

$a_0(x) = 1$ ,  $a_0 = \text{vol}(X)$ .  $a_1(x) = \frac{\tau(x)}{6}$ ,  $\tau(x)$  - scalar curvature.

### “Heat kernel” estimates:

**Theorem 2b**[JP] If the scalar curvature  $\tau(x) \neq 0$ ,  $\implies R_x(\lambda) = \Omega(\lambda^{n-2})$ .

**Global**:[JPT] If  $\int_X \tau \neq 0$ ,  $\implies R(\lambda) = \Omega(\lambda^{n-2})$ .

**Remark:** if  $\tau(x) = 0$ , let  $k = k(x)$  be the first positive number such that the  $k$ -th local heat invariant  $a_k(x) \neq 0$ . If  $n - 2k(x) > 0$ , then

$$R_x(\lambda) = \Omega(\lambda^{n-2k(x)}).$$

Similar result holds for  $R(\lambda)$ : if  $\int a_k(x) dx \neq 0$  and  $n - 2k > 0$ , then

$$R(\lambda) = \Omega(\lambda^{n-2k}).$$

**Oscillatory error term:** subtract  $[(n - 1)/2]$  terms coming from the heat trace:

$$N_x(\lambda) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \frac{a_j(x)\lambda^{n-2j}}{(4\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}-j+1)} + R_x^{osc}(\lambda)$$

*Warning:* **not** an asymptotic expansion!

Physicists: subtract the “mean smooth part” of  $N_x(\lambda)$ .

**Theorem 2c**[JP] If  $x \in X$  is not conjugate to itself along any shortest geodesic loop, then

$$R_x^{osc}(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

**Theorem 4c**[JP]  $X$  - negatively-curved. For any  $\delta > 0$

$$R_x^{osc}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right), \text{ any } n.$$

If  $n \geq 4$  then Theorem 2b,  $R_x(\lambda) = \Omega(\lambda^{n-2})$  gives a better bound for  $R_x(\lambda)$ .

**Global Conjecture:**  $X$  - negatively-curved.

For any  $\delta > 0$

$$R^{osc}(\lambda) = \Omega \left( (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right), \text{ any } n.$$