ON NODAL SETS AND NODAL DOMAINS ON S^2 AND $\mathbf{R}^2$

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Abstract. We discuss possible topological configurations of nodal sets, in particular the number of their components, for spherical harmonics on S^2. We also construct a solution of the equation $\Delta u = u$ in $\mathbf{R}^2$ that has only two nodal domains. This equation arises in the study of high energy eigenfunctions.

1. Topological structure of nodal domains

Homogeneous harmonic polynomials of degree n in three variables, when restricted to the unit sphere, become eigenfunctions of the Laplace–Beltrami operator on the sphere, with eigenvalue $\lambda = n(n+1)$ of multiplicity $2n+1$, $n = 0, 1, \ldots$. We permit ourselves the liberty of calling them *eigenfunctions of degree* n.

The zero set of an eigenfunction is called the *nodal set*. The nodal set is *non-singular* if it is a union of disjoint closed analytic curves, and the nodal set of a generic eigenfunction is non-singular [12].

The known results about topology of nodal sets on the sphere are the following [5]. The nodal set is non-empty for $n \geq 1$. A general theorem of Courant [2] implies that the nodal set of an eigenfunction of degree n consists of at most n^2+1 components. H. Lewy [10] proved that for even $n \geq 2$ the number of components is at least 2, and constructed eigenfunctions of any degree n whose nodal sets have one component for odd n and two components for even $n \geq 2$.

We call subsets X_1 and X_2 of homeomorphic topological spaces U_1 and U_2 *equivalent* if there is a homeomorphism $h : U_1 \to U_2$ such that $h(X_1) = X_2$. The ambient spaces X_1 and X_2 are usually clear from context. We notice that each homogeneous harmonic polynomial is either even or odd, so its zero set is invariant under the antipodal map of the sphere.

Theorem 1.1. *Let $0 < m \leq n$, and let $n - m$ be even. For every set of m disjoint closed curves on the sphere, whose union E is invariant with respect to the antipodal map, there exists an eigenfunction of degree n whose zero set is equivalent to E.*

The invariance with respect to the antipodal map a is a strong restriction on the possible shape of the set (see, for example, [17] for the proofs of the following facts). For each component C of the nodal set, either $a(C) = C$, or $a(C) = C'$, where C' is another component, different from C. In the former case C is called an *odd component*, and in the latter case an *even component* or an *oval*. Odd eigenfunctions have exactly one odd component of the nodal set, and even eigenfunctions have only even components in their nodal sets.

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Thus the total number of components is of the same parity as n. This generalizes Lewy's result that there are at least two components for even $n \geq 2$.

All these facts hold for *any* homogeneous polynomials in three variables restricted to the sphere with non-singular zero sets.

Topological classification of non-singular level sets of arbitrary (not necessarily harmonic) homogeneous polynomials was obtained in [14]; the author also allows some simple singularities (double points).

Remark 1.2. Theorem 1.1 gives at most n components of the nodal set for a spherical harmonic of degree n, while the best upper estimates known to the authors are $n^2 + O(n)$. These estimates can be derived from the upper estimates of the number of nodal domains, [8] (see also [1, Problem 1], and [6, 7]). Thus we obtain from the Corollary in [8] that the number of components of the nodal set of a spherical harmonic of degree n does not exceed

i) $n^2 - 2n + 2$ for even n, and
ii) $(n-1)^2 + 3$ for odd n.

For $n \leq 6$, precise estimates of the number of nodal domains were obtained in [11]. For large n, Pleijel [13] obtained an upper bound $(4+o(1))n^2/j_0^2 \approx 0.69n^2$, where j_0 denotes the smallest zero of the 0-th Bessel function. It seems difficult to determine the largest possible number of components of the nodal set of a spherical harmonic of given degree. We address this question in section 2.

In the proof of Theorem 1.1 we use a related result on topological classification of zero sets of harmonic polynomials in two variables (Theorem 1.3 below). To state Theorem 1.3 we need to fix some terminology.

A *tree* is a finite connected contractible 1-complex with at least one edge. Vertices of degree 1 are called *leaves*. A *forest* is a disjoint union of trees. An *embedded forest* is a subset of the plane which is the image of a proper embedding of a forest minus leaves to the plane.

Let u be a harmonic polynomial of two variables of degree n. Then it is easy to see that the level set $\{z : u(z) = 0\}$ is an embedded forest (indeed, existence of a cycle would contradict the maximum principle). All vertices of a forest, except the leaves, have even degrees (since the function changes sign an even number of times as we go around the vertex); there are exactly $2n$ leaves.

Theorem 1.3. *Let F be an embedded forest with $2n$ leaves and such that all its vertices in the plane are of even degrees. Then there exists a harmonic polynomial u of degree n whose zero set is equivalent to F.*

To prove Theorem 1.1, we need only the "generic case" of Theorem 1.3, when F is a union of simple curves, that is each tree of the forest has only one edge. Such forest is convenient to visualize as a *chord diagram* in the unit disc. It is obtained by an embedding of a forest to the closed unit disc such that the leaves are mapped to the boundary circle and the rest of the forest into the open disc. It is convenient to place the leaves at the roots of unity of degree $2n$ on the unit circle which we will always do.

Remark 1.4. The generic case of Theorem 1.3 (needed in the proof of Theorem 1.1) can be derived from Belyi's theorem, see e.g. [9, Theorem 2.2.9]. Using the full strength of Theorem 1.3, and some extra work, one can prove an extended version of Theorem 1.1, classifying all nodal sets, not only generic ones. Namely, one can show that any gluing of a forest with its image under the antipodal map (gluing by identifying the leafs in the natural cyclic order) is equivalent to a nodal set, and that all nodal sets arise this way.

The connection between the level sets of harmonic polynomials in the plane and the nodal sets of eigenfunctions is established by the following Lemma 1.8.

We recall that every eigenfunction of degree n can be represented in the upper hemisphere as

$$f(z) = \Re \sum_{k=0}^{n} L_k(r) a_k z^k,$$

where $z = x + iy$, is the orthogonal projection of a point of the upper hemisphere onto the equatorial plane which we identify with the complex plane, $r = |z|$ and $L_k(r) = F_n^{(k)}(\sqrt{1-r^2})$, where F_n^k are defined by

$$c_{n,k}(1-x^2)^{k/2} F_n^k(x) = \frac{d^k}{dx^k} P_n(x). \tag{1.5}$$

Here P_n is the Legendre polynomial of degree n (see, for example, [16]):

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left([x^2 - 1]^n \right), \tag{1.6}$$

and where $c_{n,k}$ is chosen so that

$$L_k(0) = F_n^k(1) = 1. \tag{1.7}$$

Let

$$p(z) = a_0 + \ldots + z^n$$

be a monic polynomial whose real part has non-singular zero set. We define a family of eigenfunctions

$$f_t(z) = \Re \sum_{k=0}^{n} L_k(r) t^{n-k} a_k z^k,$$

where $t > 0$ is a parameter.

Lemma 1.8. *If t is small enough then the zero set of f_t in the open upper hemisphere is equivalent to the zero set of $\Re p$ in the plane.*

Proof. We identify the upper hemisphere with the unit disc in the complex plane via vertical projection as before. Consider the transformed function

$$t^{-n} f_t(tz) = \Re \sum_{k=0}^{n} L_k(tr) a_k z^k.$$

When $t \to 0$ this converges to $p(z)$ in $C^1(K)$ on every disc K in the plane. Here we used the property (1.7) of Legendre polynomials. Thus for every $r_0 > 0$ there exists $t_0 > 0$ such that for $t < t_0$, the zero set of $t^{-n} f_t(tz)$ is equivalent to the zero set of $\Re p$ in the disc $\{z : |z| < r_0\}$. We fix r_0 in such a way that the portion of the zero set of $\Re p$ in $|z| < r_0$ is equivalent to its zero set in the whole plane, and thus the portion of the zero set of $\Re p$ in in $r_0 \leq |z| < \infty$ is equivalent to the zero set of $\Re z^n$.

C^1-convergence implies that the zero set of f_t with $t < t_0$ in the disc $|z| < r_0$ is equivalent to the level set of p in the plane. Now we consider the set $r_0 \leq |z| \leq 1$.

It is clear that $f_t(z) \to z^n$ in C^1 on this set, as $t \to 0$. So for t small enough the zero set of f_t in the unit disc is equivalent to the zero set of p in the plane. This proves the lemma. $\square$

Proof of Theorem 1.1. In view of Lemma 1.8, it remains to show that for every integer $m \in [1, n]$ of the same parity as n, every disjoint union E of m closed curves on the sphere that is invariant with respect to the antipodal map can be obtained by gluing together a chord diagram and its centrally symmetric chord diagram.

Case of odd m and n, $m = 2k + 1$. As we stated after Theorem 1.1, there is exactly one odd component C_0 (mapped into itself by the antipodal map). Let $C_1, C_1', \ldots, C_k, C_k'$ be the other components, such that C_j is mapped into C_j' by the antipodal map T.

First, remove the odd component C_0 from the sphere. Its complement consists of two disks D and D', and we can choose our notation so that $C_j \subset D$ and $C_j' \subset D'$ for $1 \le j \le k$. T induces a homeomorphism between the two disks, while their common boundary C_0 satisfies $T(C_0) = C_0$. Choose two antipodal points $a, b \in C_0$.

Choose a simple curve γ lying in D, so that it

i) connects a and b;
ii) intersects each C_j transversally an even number of times;
iii) the total number of intersections of γ with $\cup_{j=1}^{k} C_j$ is equal to $n - 1$.

The case $n = m = 2k + 1$, when γ intersects each C_j twice, is illustrated in Figure 1 (in the right part of this figure, γ is the upper half of the dotted circle). If $m < n$, we can construct such a γ as follows: first choose a curve $\widetilde{\gamma}$ that intersects each C_j exactly twice, for a total of $2k < n - 1$ intersections. Then adjust $\widetilde{\gamma}$ so that it intersects C_1 at two more points, and repeat the procedure $(n - 1)/2 - k$ times until the number of intersections is equal to $n - 1$.

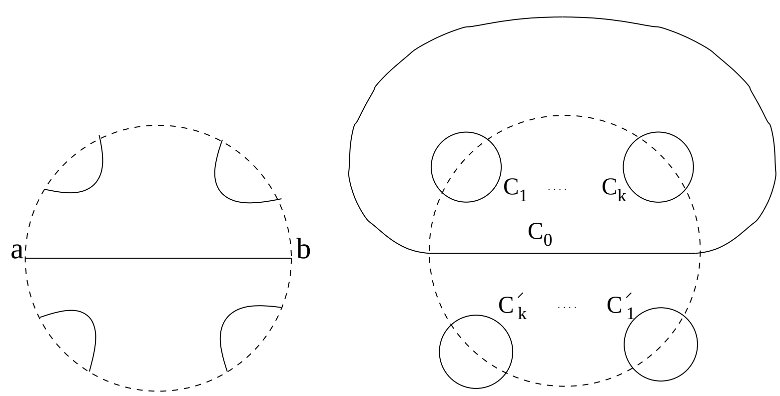

Figure 1. A chord diagram producing ovals.

Let $T(\gamma) = \gamma'$. Then $\gamma \cup \gamma' \cup \{a\} \cup \{b\}$ is a simple closed curve on S^2 (shows as a dotted circle in Fig. 1). Its complement consists of two regions G and G' satisfying $T(G) = G'$.

Let us represent G as a unit disk $\{|z| < 1\}$, so that $a = 1, b = -1$, and $T(z) = -1/\bar{z}$. The parts of curves C_j and C_j' lying inside G, together with the arc of C_0 connecting a and b inside G, form a chord diagram that has $2n-2+2 = 2n$ vertices.

The parts of curves C_j and C'_j lying inside G', together with the complementary arc of C_0 connecting a and b inside G', form *another* chord diagram, which is symmetric to the first diagram under the mapping $T(z) = -1/\bar{z}$.

An application of Lemma 1.8 finishes the proof.

Case of even n and $m = 2k$. In this case all components of the nodal set are even, and there is exactly one complementary region to our set E which is mapped by the antipodal map onto itself. One can find a closed curve α in this region so that α is invariant under the antipodal map. Clearly this α is disjoint from E. Adding it to E we can perform the construction as above, and then remove one edge corresponding to α from the chord diagram. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. We use the method of [3]. A component of the complement of the embedded forest is called a *face*. All faces are simply connected unbounded regions The degree of a face is defined as the number of connected components of its boundary. It is evident that the sum of the degrees of all faces is $2n$.

A *labeling* of a forest is a prescribing of positive numbers to the edges, so that the following condition is satisfied: for each face, the sum of these numbers over all edges on the boundary of this face equals 2π times the degree of the face.

It is easy to show that for every forest there exists a labeling. In the special case that there are no vertices (the only case needed in this paper) the labeling is unique and the label of each edge is 2π.

To define a labeling in the general case, we notice that every forest can be obtained from a forest without vertices by applying the following operation finitely many times. Choose an edge $E = [a, \infty)$ whose one extremity is ∞. Choose a point v in the interior of this edge. Then replace the part $[v, \infty)$ of E by an odd number of edges $E_1, \ldots, E_{2k+1}$ whose extremities are v and ∞. Thus E is replaced by $E_0 = [a, v] \subset E$ and $E_1, \ldots, E_{2k+1}$. We suppose that $E_0, E_1, \ldots, E_{2k+1}$ are enumerated in a natural cyclic order around v.

Now suppose that the label of E was x. In the new tree, we label E_0 by $x/2$, and then the labels of $E_1, \ldots, E_{2k+1}$ will be $x/2$ and $2\pi - x/2$, alternatively. It is clear that this gives a labeling of the new tree.

Once a labeling is prescribed, we define the "length" of a path in the forest as the sum of the labels of edges of this path.

Now we orient the edges observing the following rule: if v is a vertex and $e_1, \ldots, e_{2k}$ are all edges attached to it, listed in the natural cyclic order then of any pair e_j and e_{j+1} one edge is oriented towards v and another away from v.

It is easy to see that there are exactly two ways to orient the edges of a tree according to this rule; orientation of one edge induces orientation of the rest. We begin by choosing orientation of the edges of one tree of the forest. This induces orientation on the boundaries of each face having common boundary with this tree. So we obtain orientation of all other trees which intersect the boundary of these faces. Continuing this procedure we orient the edges of the whole forest, and in particular the boundaries of all faces.

Now we construct a ramified covering from the forest to the real line $\mathbf{R}$. We equip the real line with the spherical metric $2|dx|/(1 + |x|^2)$, so that the length of the real line is 2π. We orient the real line from $-\infty$ to ∞. Every edge of the forest will be mapped homeomorphically and respecting the orientation (that is increasing with respect to the orientations of the edge and of the real line) onto an interval of $\mathbf{R}$ whose length is the label of the edge. All leaves are mapped to ∞.

Such a continuous map ϕ is uniquely defined once the orientations of the edges and their labels are fixed. It is a ramified covering (ramified at the vertices).

Now we extend ϕ to the faces. Every boundary component of a face is mapped on the real line homeomorphically. Our choice of the orientation of each tree guarantees that the whole boundary (in $\mathbf{C}$) of each face is mapped on the real line by a covering map. Considering the boundary of the face in $\mathbf{C}$ and the real line as circles we obtain a covering of circles.

We extend it to a ramified covering of the discs, for example, with at most one ramification point in each disc (none, if the boundary map is a homeomorphism).

Thus we obtain a ramified covering of the sphere, so that the preimage of the real line is our embedded forest, and the full preimage of infinity is infinity.

We pull back the standard conformal structure via ϕ and obtain some conformal structure in the domain of ϕ. By the Uniformization theorem, there exists a homeomorphism ψ such that $\phi \circ \psi$ is holomorphic. As it is also of degree n (=half of the number of leaves of the tree), we obtain a complex analytic polynomial of degree n. The preimage of the real line is evidently equivalent to our given forest. ☐

2. Number of ovals

In this section for every $n \gg 0$, we construct examples of spherical harmonics with many ovals. All spherical harmonics in this section are assumed to be non-singular; their nodal lines don't intersect. Recall that a standard basis of spherical harmonics of degree n is given (up to proportionality constant which is unimportant for us) by functions

$$Y_n^m(\theta, \phi) = \sin^m \theta F_n^m(\cos\theta) \sin(m\phi), \qquad 0 \leq m \leq n,$$

where F_n^m was defined in (1.5). The function $(1-x^2)^{m/2} F_n^m(x)$ is proportional to the *associated Legendre function*; its zeros (aside from $x = \pm 1$) coincide with those of $F_n^m(x)$.

For a spherical harmonic Ψ, we denote by $\mathrm{ov}(\Psi)$ the number of disjoint connected components ("ovals") of its nodal set.

We prove the following

Theorem 2.1. *For any $n \gg 0$, there exist $\epsilon_n > 0$ and a rotation $R_n \in \mathrm{SO}(3)$ such that for any $\epsilon < \epsilon_n$,*

i) *For* odd n,

$$\lim_{n\to\infty} \frac{\mathrm{ov}(Y_n^{[n/2]} + \epsilon \cdot Y_n^{n-1} \circ R_n)}{n^2/4} = 1.$$

ii) *For* even n,

$$\lim_{n\to\infty} \frac{\mathrm{ov}(Y_n^{n/2} + \epsilon \cdot Y_n^{n} \circ R_n)}{n^2/4} = 1.$$

Remark 2.2. Theorem 2.1 complements results in [10], where spherical harmonics with the *smallest* possible number of ovals (and nodal domains) are constructed. Leydold conjectured in [11] that spherical harmonics $Y_n^{[n/2]}$ (that we are perturbing) have the *largest* possible number of nodal domains. However, their nodal sets are connected. *Upper* bounds on the number of nodal domains for spherical harmonics were discussed in Remark 1.2; for *irreducible* curves Harnack's bound gives better results, see [4]. It seems interesting to construct explicit examples of spherical harmonics whose nodal sets consist of a *prescribed* number of ovals between one (or two in the even degree case) and $n^2/4$.

Proof of Theorem 2.1. For convenience, we give separate proofs for different $n(\text{mod}4)$.

Case i): $n = 4k+3$. This is the easiest case.

The nodal sets of the harmonic Y_n^{2k+1} consists of

i) north and south poles ($\theta = 0, \theta = \pi$).
ii) $2k+2$ parallels $\theta = \theta_j, \theta = \pi - \theta_j$, where

$$0 < \theta_1 < \theta_2 < \ldots < \theta_{k+1} < \pi/2;$$

here $F_n^{2k+1}(\cos\theta_j) = 0$.
iii) $4k+2$ half-meridians $\phi = \pi j/(2k+1), 0 \le j < 4k+2$.

The nodal lines intersect at the poles; as well as at $(2k+2)(4k+2)$ points in the region $\{0 < \theta < \pi\}$, where two nodal lines intersect at each point. We denote the intersection points in the upper hemisphere $\{0 < \theta < \pi/2\}$ by $P_j, 1 \le j \le (k+1)(4k+2)$; and the intersection points in the lower hemisphere $\{\pi/2 < \theta < \pi\}$ by $Q_j, 1 \le j \le (k+1)(4k+2)$.

The nodal set of the harmonic Y_n^{4k+2} consists of

i) north and south poles ($\theta = 0, \theta = \pi$).
ii) the equator $\theta = \pi/2$;
iii) $8k+4$ half-meridians $\phi = \pi j/(4k+2), 0 \le j < 8k+4$.

The nodal domains of Y_n^{4k+2} are $8k+4$ sectors in the upper hemisphere: $\{0 < \theta < \pi/2; 0 < \phi < \pi/(4k+2)\}$, $\{0 < \theta < \pi/2; \pi/(4k+2) < \phi < 2\pi/(4k+2)\}$, etc. and symmetric $8k+4$ sectors in the lower hemisphere.

Choose $\tilde{R}_n$ to be the rotation with NS-axis, and with an angle ψ_n satisfying $0 < -\psi_n < \pi/(4k+2)$. Then it is easy to see that

Lemma 2.3. *There are exactly* two *nodal meridians of* $Y_n^{4k+2} \circ \tilde{R}_n$ *lying between any two nodal meridians of* Y_n^{2k+1}. *Accordingly,* $Y_n^{4k+2} \circ \tilde{R}_n$ *takes* negative *values at all the points* P_j*; and* positive *values at all the points* Q_j, $1 \le j \le (k+1)(4k+2)$.

We can next adjust $\tilde{R}_n$ slightly to get a rotation R_n so that the conclusion of Lemma 2.3 still holds, and also $Y_n^{4k+2} \circ R_n$ takes *positive* value at S and *negative* value at N.

It is then clear that if we choose ϵ_n small enough, then the nodal set of $Y_n^{2k+1} + \epsilon_n \cdot Y_n^{4k+2} \circ R_n$ will consist of

i) one "long" oval along the equator.
ii) $(k+1)(2k+1)$ "small" ovals bounding *positive* nodal domains in the upper hemisphere.
iii) $(k+1)(2k+1)$ "small" ovals bounding *negative* nodal domains in the lower hemisphere.

Altogether, the nodal set consists of $(k+1)(4k+2)+1 \sim n^2/4$ disjoint ovals, finishing the proof for $n = 4k+3$.

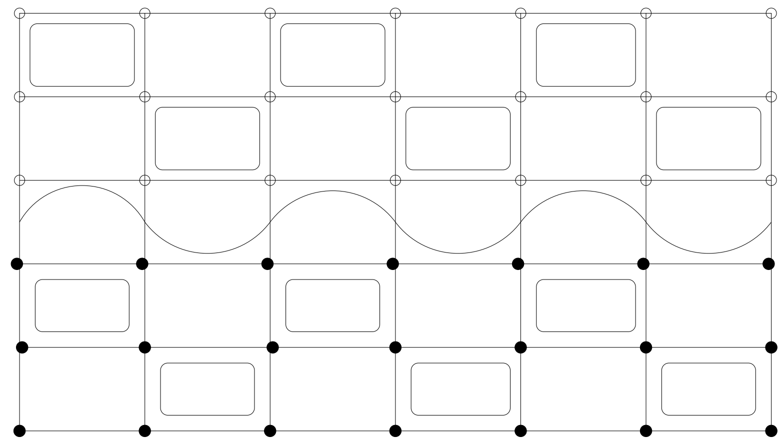

Figure 2. A perturbation in degree $4k+3=7$; $\bullet$ denotes positive sign of Y_7^6 at singular points of Y_7^3, while $\circ$ denotes negative sign of Y_7^6 at singular points of Y_7^3. Note that on S^2 all points on the top line (north pole) and the bottom line (south pole) are identified.

□

Case ii): $n = 4k+1$. The proof in this case proceeds along the same lines as in i), except we lose a linear number of ovals due to the fact that it is impossible to arrange for the perturbing harmonic to have constant sign at so many adjacent nodal crossings as for $n = 4k+3$.

Consider the spherical harmonic Y_{4k+1}^{2k}. Its nodal set consists of

i) north and south poles ($\theta = 0, \theta = \pi$).
ii) $2k$ parallels $\theta = \theta_j, \theta = \pi - \theta_j$, where

$$0 < \theta_1 < \theta_2 < \ldots < \theta_k < \pi/2;$$

here $F_n^{2k}(\cos\theta_j) = 0$.
iii) $4k$ half-meridians $\phi = \pi j/(2k), 0 \le j < 4k$.

The nodal lines intersect at the poles; $4k$ points on the equator; as well as at $(2k)(4k)$ points in the region $\{0 < \theta < \pi\}$, where two nodal lines intersect at each point. We denote the intersection points in the upper hemisphere $\{0 < \theta < \pi/2\}$ by $P_j, 1 \le j \le k{\cdot}4k$; and the intersection points in the lower hemisphere $\{\pi/2 < \theta < \pi\}$ by $Q_j, 1 \le j \le k \cdot 4k$.

The nodal set of the harmonic Y_n^{4k} consists of

i) north and south poles ($\theta = 0, \theta = \pi$).
ii) the equator $\theta = \pi/2$;
iii) $8k$ half-meridians $\phi = \pi j/(4k), 0 \le j < 8k$.

The nodal domains of Y_n^{4k} are $8k$ sectors in the upper hemisphere: $\{0 < \theta < \pi/2; 0 < \phi < \pi/(4k)\}$, $\{0 < \theta < \pi/2; \pi/(4k) < \phi < 2\pi/(4k)\}$, etc. and symmetric $8k$ sectors in the lower hemisphere.

Choose $\tilde{R}_n$ to be the rotation with NS-axis, and with an angle ψ_n satisfying $0 < -\psi_n < \pi/(4k)$. Then it is easy to see that

Lemma 2.4. *There are exactly* two *nodal meridians of* $Y_n^{4k} \circ \tilde{R}_n$ *lying between any two nodal meridians of* Y_n^{2k}. *Accordingly,* $Y_n^{4k} \circ \tilde{R}_n$ *takes* negative *values at all the points* P_j*; and* positive *values at all the points* Q_j, $1 \le j \le k \cdot 4k$.

We can next adjust $\tilde{R}_n$ slightly to get a rotation R_n so that the conclusion of Lemma 2.4 still holds, and also $Y_n^{4k} \circ R_n$ takes

i) *Positive* value at S, and *negative* value at N.
ii) *Positive* values at equatorial intersection points for which $0 < \phi \le \pi$, and *negative* values at equatorial intersection points for which $\pi < \phi \le 2\pi$

It is then clear that if we choose ϵ_n small enough, then the nodal set of $Y_n^{2k} + \epsilon_n \cdot Y_n^{4k} \circ R_n$ will consist of

i) one "long" oval along the equator.
ii) *at least* $k \cdot 2k$ "small" ovals bounding *positive* nodal domains in the upper hemisphere.
iii) *at least* $k \cdot 2k$ "small" ovals bounding *negative* nodal domains in the lower hemisphere.

Actually, there will be some additional small nodal domains in upper and lower hemisphere, but we shall ignore them.

Altogether, the nodal set consists of at least $4k^2 + 1 \sim n^2/4$ disjoint ovals, finishing the proof for $n = 4k + 1$. □

Case iii): $n = 4k, n = 4k + 2$.

Let $n = 2m$. We shall be perturbing the spherical harmonics Y_{2m}^m by (small multiples of) Y_{2m}^{2m} which is proportional to $\sin(2m\phi)$. Nodal meridians of Y_{2m}^m are the meridians

$$\mathcal{N}_1 = \{\phi = 0, \phi = \pi/m, \phi = 2\pi/m, \ldots, \phi = \pi, \phi = \pi(m+1)/m, \ldots\},$$

while those of $Y_{2m}^{2m} \circ R(-\delta)$ (here R denotes a rotation around NS-axis; we shall chose $\delta \ll 1/2m$) have the form

$$\mathcal{N}_2 = \{\phi = \delta, \phi = \pi/(2m) + \delta, \phi = 2\pi/(2m) + \delta, \ldots\}$$

It is easy to show that

Lemma 2.5. *For small enough* δ *there will be* two *meridians from* $\mathcal{N}_2$ *between any two meridians of* $\mathcal{N}_1$.

Accordingly, one can arrange for $Y_{2m}^{2m} \circ \tilde{R}_{2m}$ to have constant (say, positive) sign at all double intersections of the nodal lines of Y_{2m}^m different from N and S. By adjusting the rotation slightly, we can arrange for $Y_{2m}^{2m} \circ \tilde{R}_{2m}$ to be nonzero at N and S as well (and to have positive sign there as well).

Accordingly, for small enough ϵ, the nodal set of the spherical harmonic $Y_{2m}^m + \epsilon Y_{2m}^{2m} \circ \tilde{R}_{2m}$ will have $m(m+1) \sim n^2/4$ ovals as claimed, finishing the proof of case ii) of Theorem 2.1.

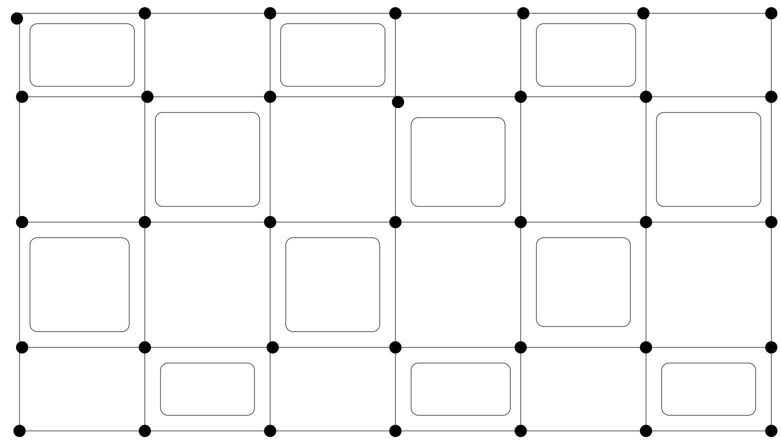

Figure 3. A perturbation in degree $2m = 6$; • denotes positive sign of Y_6^6 at singular points of Y_6^3. Note that on S^2 all points on the top line (north pole) and the bottom line (south pole) are identified.

□

3. Eigenfunctions in $\mathbf{R}^2$ with two nodal domains

Let M be a compact surface, and let Δ be the Laplace-Beltrami operator for a metric on M, and let $\Delta u_k = \lambda_k u_k$ be a sequence of eigenfunctions. Let $x_k \in M$ be a point of supremum of u_k, and let B_k be a geodesic ball centered at x_k with radius $C/\sqrt{\lambda_k}$. Blow up B_k to the unit disk in $\mathbf{R}^2$, and let $\tilde{u}_k/\sup u_k$ be the eigenfunction after that change of variables. Then (after passing to a subsequence) $\tilde{u}_k$ converges to a solution

$$\Delta \tilde{u} = \tilde{u}, \qquad |\tilde{u}| < 1. \tag{3.1}$$

Thus the nodal structure of $\tilde{u}$ models fine local nodal structure of u_k, e.g. if one is able to prove that u has infinite number of isolated critical points, then one can expect their number to be unbounded also for the sequence u_k. We guess here that any solution of 3.1 on $\mathbf{R}^2$ has infinite number of critical points. However, a bit stronger statement on the infinite number of nodal domains of $\tilde{u}$ turns not to be true.

Somewhat surprisingly, we prove the following

Theorem 3.2. *There exist solutions of* (3.1) *with exactly* two *nodal domains.*

Proof of Theorem 3.2. The proof is inspired by a related construction in [10]. Take a function $f(x, y)$ defined in polar coordinates $x = r\cos\theta, y = r\sin\theta$ by

$$f = J_1(r)\sin\theta.$$

Then zeros of f consist of the x-axis, together with an infinite union of concentric circles

$$r = j_k, k = 1, 2, \ldots$$

where j_k denotes the $k - th$ zero of J_1.

We shall use the following standard result (see e.g. [15, Thm 1.6]):

Lemma 3.3. *There exists* $C > 0$ *such that*

$$\inf_k |j_{k+1} - j_k| > C.$$

An easy consequence of Lemma 3.3 is the following

Lemma 3.4. *Let* $0 < \delta_2 < \delta_1 < C/2$*; define function* $g(x, y)$ *by the formula*

$$g(x, y) := f(x - \delta_1, y - \delta_2).$$

Then $g(j_k, 0) = (-1)^{k-1}$ *and* $g(-j_k, 0) = (-1)^k$.

Finally, it is easy to see that the function

$$h(x, y) := f(x, y) + \epsilon \cdot g(x, y)$$

will have the properties required in Theorem 3.2 for ϵ small enough. □

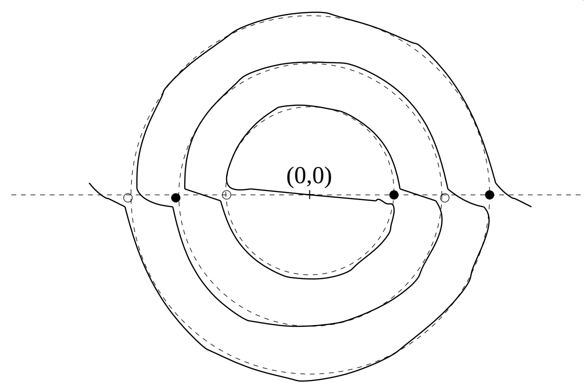

Figure 4. A function in $\mathbf{R}^2$ with two nodal domains. Dashed line denotes the nodal set of $f = J_1(r)\sin\theta$, solid line denotes the nodal set of the perturbed eigenfunction h. • denotes positive sign of g at singular points of f, ∘ denotes negative sign of g at singular points of f.

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