Eigenfunctions of laplacian

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September 13, 2007

The *Laplacian* Δ of a function f is given by

$$\Delta f = \text{div}(\text{grad} f).$$

An *eigenfunction* ϕ with *eigenvalue* $\lambda \geq 0$ satisfies

$$\Delta f + \lambda f = 0.$$

Example 1: $\mathbf{R}^2$.

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Periodic eigenfunctions on the 2-torus $\mathbf{T}^2$: $f(x \pm 2\pi, y \pm 2\pi) = f(x, y)$. They are

$$\sin(m \cdot x + n \cdot y), \cos(m \cdot x + n \cdot y), \quad \lambda = m^2 + n^2.$$

Fact: any square-integrable function $F(x, y)$ on $\mathbf{T}^2$ (s.t. $\int_{\mathbf{T}^2} |F(x, y)|^2 dxdy < \infty$), can be expanded into Fourier series,

$$F = \sum_{m,n=-\infty}^{+\infty} a_{m,n} \sin(mx + ny) + b_{m,n} \cos(mx + ny).$$

Example 2: sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Spherical coordinates: $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$, where $x = \sin\phi\cos\theta, y = \sin\phi\sin\theta, z = \cos\phi$.

$$\Delta f = \frac{1}{\sin^2\phi} \cdot \frac{\partial^2 f}{\partial\theta^2} + \frac{\cos\phi}{\sin\phi} \cdot \frac{\partial f}{\partial\phi} + \frac{\partial^2 f}{\partial\phi^2}$$

Eigenfunctions are called *spherical harmonics*:

$$Y_l^m(\phi, \theta) = P_l^m(\cos\phi)(a\cos(m\theta) + b\sin(m\theta)).$$

Here $\lambda = l(l+1)$; $P_l^m, |m| \le l$ is *associated Legendre function*,

$$P_l^m(x) = \frac{(-1)^m}{2^l \cdot l!}(1 - x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} \left((x^2 - 1)^l\right).$$

Any square-integrable function F on S^2 can be expanded in a series of spherical harmonics.

The same is true on any *compact* (e.g. closed and bounded) curved surface S: any square-integrable function F on S can be expanded in a series of eigenfunctions of Δ.

Similar results hold in higher dimensions, and for domains with boundary.

Applications: Solving partial differential equations like *heat equation* $\partial u(x,t)/\partial t = c\cdot\Delta_x u(x,t)$ and *wave equation* $\partial^2 u(x,t)/\partial t^2 = c\cdot\Delta_x u(x,t)$.

Stationary solutions of *Schrödinger equation* or “pure quantum states.”

Inverse problems: suppose you know some eigenvalues and eigenfunctions; describe the domain S (related problems appear in radar/remote sensing, x-ray/MRI, oil/gas/metal exploration etc).

Mathematical Problems:

- Determine *the smallest* $\lambda > 0$ for a given surface S (its “bass note”), and the corresponding eigenfunction.
- “Can you hear the shape of a drum:” can two different domains S have the same *spectrum*, e.g. the collection of all $\{0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\}$?
- Count the eigenvalues: $N(T) = \#\{\lambda_j < T\}$. How fast does $N(T)$ grow as $T \to \infty$?

Example: 2-torus $\mathbf{T}^2$: $\lambda_{m,n} = m^2 + n^2$. Let $T = R^2$. Then

$$N(R^2) = \#\{(m,n) : m^2 + n^2 < R^2\} =$$
$$\#\{(m,n) : \sqrt{m^2+n^2} < R\}.$$

How many lattice points are inside the circle of radius R? Leading term is given by the *area*:

$$N(R^2) = \pi R^2 + E(R), \qquad (1)$$

where $E(R)$ is the *remainder*.

Question: How big is $E(R)$? Conjecture (Hardy): for any $\delta > 0$,

$$E(R) < C(\delta) \cdot R^{1/2+\delta}, \qquad as\ R \to \infty.$$

Best known estimate (Huxley, 2003):

$$E(R) < C \cdot R^{131/208} (\log R)^{2.26}.$$

Note: $131/208 = 0.629807...$.

An analogue of (1) holds for very general domains; it is called *Weyl's law* (Weyl, 1911). Much less is known about $E(R)$.

Questions about eigenfunctions: Let $\Delta f + \lambda f = 0$, λ-large ("high energy").

• Where is f concentrated, i.e. describe $\{(x, y) : |f(x, y)| \ is\ large\}$?

Ex: on some domains with boundary, "whispering gallery" eigenfunctions concentrate near the boundary.

• *Nodal sets*: study $\{(x, y) : f(x, y) = 0\}$. This will generally be a *curve*, or a union of curves. First pictures: *Chladni plates* (E. Chladni, 18th century; see google video links on my home www-page).

Ex: On $\mathbf{T}^2$, function $f(x, y) = \sin(mx) \sin(ny)$ vanishes on a rectangular *grid*:

$$\{(x, y) : x = \pi j/m,\ or\ y = \pi k/n\}.$$

In general, much less is known about nodal sets.