Nodal sets of eigenfunctions of Laplacian

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D. Jakobson and D. Mangoubi. "Tubular neighborhoods of nodal sets and Diophantine approximation." arXiv:0707.4045, submitted for publication.

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$X^n, n \geq 2$ - compact. Δ - Laplacian. Spectrum: $\Delta\phi_i + \lambda_i\phi_i = 0, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$

$\phi = \phi_\lambda$-eigenfunction. *Nodal set* $\mathcal{N}(\phi) := \{x \in X : \phi(x) = 0\}$. *Critical set* $\Sigma(\phi) := \{x \in X : \nabla\phi(x) = 0\}$ (gradient vanishes). $\Sigma_0(\phi) = \mathcal{N}(\phi) \cap \Sigma(\phi)$.

$\Sigma_0(\phi)$ has locally finite $(n-2)$-dimensional Hausdorff measure (Hardt, M. and T. Hoffmann-Ostenhof, Nadirashvili, 1999). The set $\mathcal{N}(\phi) \setminus \Sigma_0(\phi)$ is $(n-1)$-dimensional submanifold of X.

Theorem 1 (Donnelly, Fefferman, 1988). (X, g) real-analytic, then $\exists c_1, c_2 > 0$ s.t.

$$c_1 \leq \frac{\mathcal{H}^{n-1}(\mathcal{N}(\phi_\lambda))}{\sqrt{\lambda}} \leq c_2. \tag{1}$$

Conjecture (Yau). The same estimate should hold for smooth metrics g. In dimension 2,

$c_1\sqrt{\lambda} \le \mathcal{H}^{n-1}(\mathcal{N}(\phi_\lambda)) \le c_2\lambda^{3/4}$ (Brüning, 1978; D-F, 1990).

Conjecture holds for *random linear combinations* of eigenfunctions (Berard, Rudnick-Wigman).

$\mathcal{N}(\phi_\lambda)$ is $(C/\sqrt{\lambda})$-dense: $\exists C > 0$ s.t.

$$\forall y \in X, \quad B(y, C/\sqrt{\lambda}) \cap \mathcal{N}(\phi_\lambda) \neq \emptyset.$$

$C/\sqrt{\lambda}$- "wavelength."

What about *neighborhoods* of nodal sets? A δ-neighborhood of $\mathcal{N}(\phi_\lambda)$ is the set

$$T(\lambda, \delta) := \{x \in X : \mathrm{dist}(x, \mathcal{N}(\phi_\lambda)) < \delta\}.$$

Theorem 2 (J-Mangoubi, 2007). (M, g)-real-analytic. $\exists c_1, c_2, c_3 > 0$ s.t.

$$\forall \delta < \frac{c_3}{\sqrt{\lambda}}, \quad c_1 \le \frac{\mathrm{vol}(T(\lambda, \delta))}{\delta\sqrt{\lambda}} \le c_2.$$

Proof: in dimension 2 (M. Sodin); easier than $n > 2$. Cover X by "small" cubes A_j of size $\delta/3$ and "large" cubes B_j of size δ; every cube intersects a bounded number of other cubes.

$$\text{area}T(\lambda,\delta) \le C \sum_{A_j \cap \mathcal{N}(\phi_\lambda) \neq \emptyset} \text{area}(A_j).$$

Q-small cube, Q_1-concentric large cube, $Q \cap \mathcal{N}(\phi_\lambda) \neq \emptyset$. Two cases:
(i) all connected components of $\mathcal{N}(\phi_\lambda) \cap Q$ don't intersect $\partial Q'$;
(ii) some connected component of $\mathcal{N}(\phi_\lambda) \cap Q$ intersects $\partial Q'$.

In case (ii), length$(\mathcal{N}(\phi_\lambda) \cap Q') \ge \delta/3$. So, number of small cubes is $\ll$ length$(\mathcal{N}_\lambda)/\delta \le C\sqrt{\lambda}/\delta$, hence the sum of their areas is $\le C\sqrt{\lambda}\delta$ by [D-F].

In case (i), Q' contains at least one nodal domain D of ϕ_λ, whose area is $\ge C/\lambda$ by Faber-

Krahn inequality. By the isoperimetric inequality, the length$(\partial D) \geq C/\sqrt{\lambda} \geq C\delta$. By the previous argument, the sum of the areas of cubes of type (i) is $\leq C\sqrt{\lambda}\delta$. QED

For $n \geq 3$, the proof is more difficult, involves carefully adapting the proof in [D-F].

Application to approximation by nodal sets: How fast can one approximate a "typical" point on X by nodal sets of eigenfunctions?

Number theory motivation: $X = [0, \pi]$ with Dirichlet b.c. $\phi_k(x) = sin(kx), \lambda_k = k^2$. Nodal set: $\mathcal{N}(\phi_k) = \{\pi j/k, 0 \leq j \leq k\}$. Approximation (after rescaling by π) reduces to approximating real numbers by rational numbers.

Prop. Let $x \in [0, 1]$, p/q-continued fraction of x. Then $|x - p/q| < 1/q^2$. Also, $\forall \epsilon > 0$ and $\forall C > 0$,

$$\text{meas}\,\{x : \left|x - \frac{p_j}{q_j}\right| < \frac{C}{q_j^{2+\epsilon}}, q_1 < q_2 < \dots\} = 0.$$

Want: analogue of the previous estimate for nodal sets.

Proof for S^1: Fix $C, \epsilon > 0$. Then

$$\text{meas } A_q := \text{meas } \left\{ x : \left| x - \frac{p}{q} \right| < \frac{C}{q^{2+\epsilon}} \right\} = \frac{2C}{q^{1+\epsilon}}.$$

Then since $\sum_q \text{meas}(A_q) < \infty$, by Borel-Cantelli lemma

$$\text{meas}\{x : x \in A_q \ for \ inf. \ many \ q\} = 0.$$

On a manifold: fix a basis $\{\phi_\lambda\}$ of $L^2(X)$.

Theorem 3 (J-Mangoubi, 2007): (X, g) real-analytic. Then $\forall C > 0, \forall \epsilon > 0$,

$$\text{vol } \{x \in X : B\left(x, \frac{C}{\lambda^{(n+1+\epsilon)/2}}\right) \cap \mathcal{N}(\phi_\lambda) \neq \emptyset$$

$$for \ inf. \ many \ \lambda\} = 0.$$

Proof. Let $\delta(\lambda) = C/\lambda^{(n+1+\epsilon)/2}$ in Theorem 2. Then by Theorems 1 and 2,

$$\frac{c_1}{\lambda^{(n+\epsilon)/2}} \leq \text{vol}\, T(\lambda, \delta(\lambda)) \leq \frac{c_2}{\lambda^{(n+\epsilon)/2}}$$

Weyl's law $\Rightarrow \lambda_k \sim ck^{2/n}$ as $k \to \infty$. So,

$$\sum_{\lambda} \text{vol}\, T(\lambda, \delta(\lambda)) \leq \sum_{k=1}^{\infty} \frac{C}{k^{1+\epsilon/n}} < \infty.$$

Application of Borel-Cantelli lemma finishes the proof.

Remark. For *smooth* metrics in dimension 2, it follows from [D-F], 1990 that

$$\text{area}\, \{x \in X : B\left(x, \frac{C}{\lambda^{7/4+\epsilon}}\right) \cap \mathcal{N}(\phi_\lambda) \neq \emptyset$$

$$for\ inf.\ many\ \lambda\} = 0.$$

Further problems:

- Study curvature of $\mathcal{N}(\phi)$.
- Determine the rate of approximation by nodal

set for a typical point $x \in X$, i.e. find $E = \sup b > 0$ s.t.

$$\text{vol}\{x \in X : d(x, \mathcal{N}(\phi_\lambda)) < C/\lambda^b \ inf.\ often\} > 0.$$

Theorem 3 $\Rightarrow$ on real-analytic (X, g), $1/2 \leq E \leq (n+1)/2$.

Conjecture. If we can separate variables on X (e.g. completely integrable systems: surface of revolution, Liouville torus etc), then $E = 1$. Explanation: if $\dim X = 1$, then $E = 2$ (continued fractions). For separable systems, after a change of coordinates nodal sets form a *grid* of hyperplanes, so approximation reduces to 1-dimensional problems. Can prove for the case of generic rectangular torus with Dirichlet b.c.

Nodal domain of ϕ_λ is a connected component of $X \setminus \mathcal{N}(\phi_\lambda)$.

Theorem (Courant). Let $\Delta\phi_k + \lambda_k \phi_k = 0, 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ Then the number of nodal domains of ϕ_k is $\leq k+1$.

The constant was improved by Pleijel (1956).

Examples with *few* nodal domains: Courant, $[0,1] \times [0,1]$ with Dirichlet b.c; $\mathbf{T}^2, \sin(nx + y), n \to \infty$; S^2, H. Lewy (1977): 2 nodal domains for spherical harmonics of *odd* degree, and 3 nodal domains for for spherical harmonics of *even* degree.

Random spherical harmonic has *disjoint* nodal lines (Neuheisel, 1994). Also, $\mathcal{N}(\phi_\lambda)$ is invariant under the antipodal map on S^2.

Theorem 4 (Eremenko-J-Nadirashvili, 2006). Let $0 < m \leq n$, and let $n - m$ be even. For every set of m disjoint closed curves on the sphere, whose union E is invariant with respect to the antipodal map, there exists an spherical harmonic of degree n whose zero set is equivalent (homeomorphic) to E.

Remark: It is interesting to determine the *smallest* degree n for which a given configuration of m nodal lines appears. Can probably expect $m \sim \sqrt{n}$, since e.g. *random* spherical harmonic of degree n has $\sim cn^2$ nodal domains (Nazarov-Sodin, 2006).

Proof uses a related result about nodal sets of *harmonic polynomials*. A nodal set $\mathcal{N}(P)$ is an embedded *forest* in $\mathbf{R}^2$ (existence of a cycle would contradict maximum principle). Also, all finite vertices have *even degrees* (count sign changes of P as you go around a vertex).

Theorem 5 (E-J-N, 2006). Let F be an embedded forest with $2n$ leaves and such that all its vertices in the plane are of even degrees. Then there exists a harmonic polynomial P of degree n whose zero set is equivalent to F.

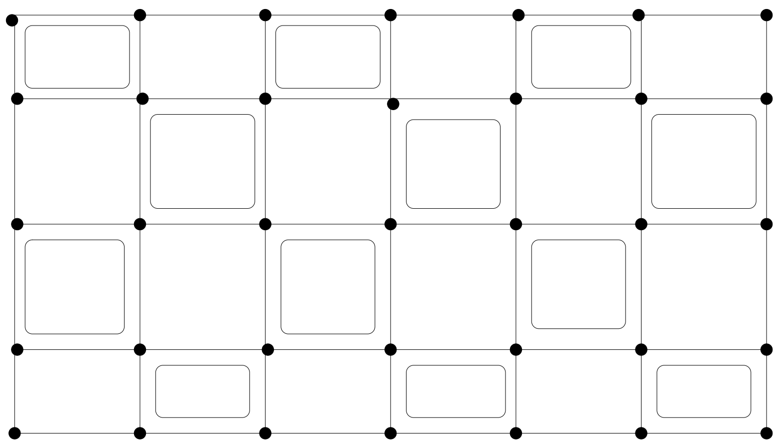
Theorem 4 follows from a special case when every tree has only *one* edge. This case can be derived from Belyi's theorem.

Theorem 5 $\Rightarrow$ Theorem 4: choose a harmonic whose nodal set in the upper hemisphere is equivalent to the nodal set of P.

Proof of Theorem 5 uses methods due to Eremenko and Gabrielov.

Question: ϕ-spherical harmonic of degree n. How many *disjoint components* can $\mathcal{N}(\phi)$ have? In standard examples, $\mathcal{N}(\phi)$ is connected.

There are examples (E-J-N, 2006) with $\sim n^2/4$ disjoint components. Example (even n):



$Y_6^3 + \epsilon Y_6^6 \circ R$; • denotes positive sign of $Y_6^6 \circ R$ at singular points of Y_6^3.

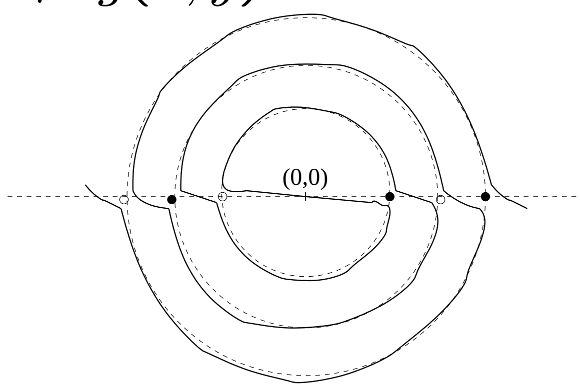
Function on a wavelength scale.

Let $\Delta\phi + \lambda\phi = 0$. After a (local) change of variables $y = x\sqrt{\lambda}$, we get a function ψ satisfying $\Delta\psi + \psi = 0$, where Δ is the planar

Laplacian. Nodal structure of ψ models that of ϕ on a scale of several wavelengths.

Theorem 6 (E-J-N, 2006). There exists a solution ψ with exactly *two* nodal domains in the whole $\mathbf{R}^2$.

Proof. In polar coordinates $x = r\cos\theta, y = r\sin\theta$, let $f(x,y) := J_1(r)\sin\theta$ and let $g(x,y) := f(x-\delta_1, y-\delta_2)$, where $0 < \delta_2 < \delta_1$ are small. Then $f(x,y) + \epsilon g(x,y)$ has two nodal domains.



$\bullet$ denotes positive sign of g at singular points of f, $\circ$ denotes negative sign of g at singular points of f.