Estimates from below for the spectral function and for the remainder in local Weyl's law

D. Jakobson (McGill)
email: jakobson@math.mcgill.ca
and
I. Polterovich (Université de Montréal)
email: iossif@dms.umontreal.ca

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$X^n, n \geq 2$ - compact.

Δ - Laplacian, $\Delta\phi_i + \lambda_i\phi_i = 0$ - spectrum.

$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$

Spectral function: Let $x, y \in X$.

$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i}\leq\lambda} \phi_i(x)\phi_i(y)$

If $x = y$, let $N_{x,y}(\lambda) := N_x(\lambda)$.

Weyl's law: $N(\lambda) = C_n V \lambda^n + R(\lambda), R(\lambda) = O(\lambda^{n-1})$

Local Weyl's law:

$N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y;$

$N_x(\lambda) = C_n\lambda^n + R_x(\lambda), \qquad R_x(\lambda) = O(\lambda^{n-1});$

$R_x(\lambda)$ - local remainder. We study *lower* bounds for $N_{x,y}(\lambda)$ and $R_x(\lambda)$.

Notation: $f_1(\lambda) = \Omega(f_2(\lambda)), f_2 > 0$ iff

$\limsup_{\lambda\to\infty} |f_1(\lambda)|/f_2(\lambda) > 0$.

Theorem 1 If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

On-diagonal ($x = y$):

Theorem 2 If the scalar curvature $\tau(x) \neq 0$, then

$$R_x(\lambda) = \Omega(\lambda^{n-2}).$$

Also, if X has no conjugate points, then

$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

Remark: if $\tau(x) = 0$, let k be such that $u_k(x, x)$ is the first nonvanishing local heat invariant ($u_1(x, x) = \frac{\tau(x)}{6}$). Then $R_x(\lambda) = \Omega(\lambda^{n-2\kappa_x})$.

Negative curvature. Suppose sectional curvature satisfies

$$-K_1^2 \leq K(\xi, \eta) \leq -K_2^2$$

Theorem (Berard): $R_x(\lambda) = O(\lambda^{n-1}/\log \lambda)$

Conjecture (Randol): On a negatively-curved surface, $R(\lambda) = O(\lambda^{\frac{1}{2}+\epsilon})$. Randol proved an integrated (in λ) version for $N_{x,y}(\lambda)$

Thermodynamic formalism: G^t - geodesic flow on SX. $\xi \in SM$, $U(\xi)$ - unstable subspace of $T_\xi SM$ for for G^t.

Sinai-Ruelle-Bowen potential $\mathcal{H} : SM \to \mathbf{R}$:

$$\mathcal{H}(\xi) = \frac{d}{dt}\Big|_{t=0} \ln \det dG^t|_{U(\xi)}$$

Topological pressure $P(f)$ of a Hölder function $f : SX \to \mathbf{R}$ satisfies

$$\sum_{l(\gamma)\leq T} l(\gamma) \exp\left[\int_\gamma f(\gamma(s), \gamma'(s))ds\right] \sim \frac{e^{P(f)T}}{P(f)}.$$

γ - geodesic of length $l(\gamma)$. $P(f)$ is defined as

$$P(f) = \sup_\mu \left(h_\mu + \int f d\mu\right),$$

μ is G^t-invariant, h_μ - (measure-theoretic) entropy.

Ex 1: $P(0) = h$, h - **topological entropy** of G^t.

Ex. 2: $P(-\mathcal{H}) = 0$. The *equilibrium measure* (attaining the supremum) for $\mathcal{H}$ is the Liouville measure μ_L on SX, thus $h_{\mu_L} = \int_{SX} \mathcal{H} d\mu_L$.

Theorem 3. If X is negatively-curved then for any $\delta > 0$ and $x \neq y$

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

Theorem 4. X - negatively-curved. For any $\delta > 0$

$$R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

If $n \geq 4$ then Theorem 2, $R_x(\lambda) = \Omega(\lambda^{n-2})$ gives a better bound. The power of the logarithm $\frac{P(-\mathcal{H}/2)}{h}$ is $\geq \frac{K_2}{2K_1} > 0$, so

$$K = -1 \;\Rightarrow R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{1}{2} - \delta}\right)$$

Karnaukh, $n = 2$: estimate above + weaker estimates in variable negative curvature.

Proofs, $x \neq y$: Theorems 1 and 3.

Wave kernel on X:

$$e(t,x,y) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t)\phi_i(x)\phi_i(y),$$

fundamental solution of the wave equation $(\partial^2/\partial t^2 - \Delta)e(t,x,y) = 0$, $e(0,x,y) = \delta(x-y)$, $(\partial/\partial t)e(0,x,y) = 0$.

$\psi \in C_0^\infty([-1,1])$, even, monotone decreasing on [0,1], $\psi \geq 0$, $\psi(0) = 1$. Fix $\lambda, T \gg 0$, consider the function

$$(1/T)\psi(t/T)\cos(\lambda t).$$

We let

$$k_{\lambda,T}(x,y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) e(t,x,y) dt$$

Pretrace formula. If X has no conjugate points, let $E(t,x,y)$ be the wave kernel on M, the universal cover of X. Then for $x, y \in X$, we have

$$e(t,x,y) = \sum_{\omega \in \Gamma = \pi_1(X)} E(t,x,\omega y)$$

Given $x, y \in M$, define $K_{\lambda,T}(x,y)$ by

$$K_{\lambda,T}(x,y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) E(t,x,y) dt.$$

Then for $x, y \in X$

$$(*)\ k_{\lambda,T}(x,y) = \sum_{\omega \in \Gamma} K_{\lambda,T}(x,\omega y)$$

The following lemma is used in the proofs:

Lemma 5 If $N_{x,y}(\lambda) = o(\lambda^a (\log \lambda)^b))$, $a > 0, b > 0$ then

$$k_{\lambda,T}(x,y) = o(\lambda^a (\log \lambda)^b)).$$

Leading term asymptotics.

Hadamard Parametrix for $E(t,x,y) \Rightarrow$

Proposition 6 Let $x \neq y \in M, r = d(x,y)$. Then $K_{\lambda,T}(x,y)$ satisfies as $\lambda \to \infty$:

$$K_{\lambda,T}(x,y) = Q_1 \lambda^{\frac{n-1}{2}} \frac{\psi(r/T)}{T\sqrt{g(x,y) r^{n-1}}} \sin(\lambda r + \theta_n) +$$

$$O(\lambda^{\frac{n-3}{2}}) + \exp(O(T)).$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates $\theta_n = (\pi/4)(3 - (n \bmod 8))$ and $Q_1 \neq 0$.

Proof by contradiction: Assume $N_{x,y}(\lambda)$ is small. Lemma 5 $\Rightarrow k_{\lambda,T}(x,y)$ is small.

Use pretrace formula and Proposition 6 to show that $k_{\lambda,T}(x,y)$ is large. Contradiction!

Proof of Theorem 1 Assume that $N_{x,y}(\lambda) = o(\lambda^{\frac{n-1}{2}})$. Lemma 5 $\Rightarrow$ $k_{\lambda,T}(x,y) = o(\lambda^{\frac{n-1}{2}})$.

$x, y \in X$ - not conjugate along any shortest geodesic, $\Rightarrow$ finitely many shortest geodesics of length $r = d(x,y)$; no geodesics from x to y of length $l \in]r, r+\epsilon]$, some $\epsilon > 0$.

Let $T = r + \epsilon/2$. Pretrace formula (*) and Proposition 6 $\Rightarrow$

$$k_{\lambda,T}(x,y) = Q\lambda^{\frac{n-1}{2}} \sum_{r_\omega = r} \sin(\lambda r_\omega + \theta_n) + O(\lambda^{\frac{n-3}{2}}),$$

where $Q \neq 0$.

Choose a sequence $\lambda_k \to \infty$ such that

$$|\sin(\lambda_k r + \theta_n)| > \nu > 0$$

Contradiction. Q.E.D.

Proof of Theorem 3. Assume for contradiction that for some $\delta > 0$,

$$N_{x,y}(\lambda) = o\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right).$$

Lemma 5 implies a similar bound for $k_{\lambda,T}(x,y)$.

Proposition 6 $\Rightarrow k_{\lambda,T}(x,y) =$

$$Q\lambda^{\frac{n-1}{2}} \sum_{r_\omega < T} \frac{\psi(\frac{r_\omega}{T})}{\sqrt{g(x,\omega y) r_\omega^{n-1}}} \sin(\lambda r_\omega + \theta_n)$$

$+O(\lambda^{\frac{n-3}{2}}) \exp(O(T))$.

Consider the sum

$$S_{x,y}(T) = \sum_{r_\omega \leq T} \frac{1}{\sqrt{g(x,\omega y)\, r_\omega^{n-1}}}$$

It follows from results of Parry and Pollicott that

Theorem 7 As $T \to \infty$,

$$S_{x,y}(T) \geq C_0 e^{P\left(-\frac{\mathcal{H}}{2}\right) \cdot T}$$

Here $P\left(-\frac{\mathcal{H}}{2}\right) \geq (n-1)K_2/2$.

Suppose $n \neq 3 (\mathrm{mod}\ 4)$. Then $\theta_n \neq 0 (\mathrm{mod}\ \pi)$.

Dirichlet box principle $\Rightarrow \exists \lambda$ so that

$$\sin(\lambda r_\omega + \theta_n) > \nu > 0, \ \forall \omega : r_\omega \leq T.$$

(λr_ω close to $2\pi\mathbf{Z}$). This combined with Theorem 7 contradicts Lemma 5. Q.E.D.

For Dirichlet principle need

$$T \asymp \ln \ln \lambda.$$

So, get logarithmic improvement in Theorem 3 compared with Theorem 1.

If $n = 3 (\text{mod } 4)$ then $\theta_n = 0 (\text{mod } \pi)$. Need a separate argument to establish that

$$\exists \lambda : \sin(\lambda r_\omega) > \frac{\nu}{T}, \qquad \forall \omega : \frac{T}{A} \le r_\omega \le T.$$

This combined with Theorem 7 contradicts Lemma 5 and proves Theorem 3 in all dimensions. Q.E.D.

On-diagonal case, $x = y$. Theorems 2 is proved by an easy heat kernel argument. Proof of Theorem 4 uses Theorem 7 and an *on-diagonal* counterpart of Proposition 6. The 0-th term of the wave parametrix on the diagonal cancels out with the main term in the Weyl's law.

Proof of Theorem 7

Step 1: From vertical to unstable subspaces. $x, y \in M$, γ - geodesic from x to y. $\xi = (x, \gamma'(0))$. $Vert(\xi) \in T_\xi SM$ - vertical subspace (tangent vectors to the unit sphere in $T_x M$); $U(\xi) \in T_\xi SM$ - unstable subspace at ξ.

Lemma 8.

$$\sqrt{g(x,y) r^{n-1}} < C \cdot \det dG^r|_{U(\xi)} = C \cdot Jac_{U(\xi)} G^r$$

Proof: $\sqrt{g(x,y) r^{n-1}} < C \cdot Jac_{Vert(\xi)} G^r$. As $r \to \infty$,

$$\text{Dist}[DG^r(Vert(\xi)), DG^r(U(\xi))] \le Ce^{-\alpha r}$$

by properties of Anosov flows, hence

$$\frac{Jac_{Vert(\xi)} G^r}{Jac_{U(\xi)} G^r}$$

remains bounded as $r \to \infty$. Q.E.D.

Let $\gamma_\omega(s), 0 \le s \le r_\omega$ - geodesic from x to ωy, $\xi(s,\omega) := (\gamma_\omega(s), \gamma'_\omega(s)) \in SM$, and $\xi(\omega) := (x, \gamma'_\omega(0))$. By definition of SRB measure $\mathcal{H}$,

$$\ln Jac_{U(\xi(\omega))} G^{r_\omega} \asymp \int_0^{r_\omega} \mathcal{H}(\xi_j(s,\omega))ds.$$

Corollary 9.

$$S_{x,y}(T) \ge C \sum_{r_\omega \le T} \exp\left(\frac{-1}{2} \int_0^{r_\omega} \mathcal{H}(\xi(s,\omega))ds\right)$$

Step 2: From loops to closed geodesics.

Lemma 10. Geodesics γ_1, γ_2 both start at $x \in M$, and $\text{dist}_M(\gamma_1(r), \gamma_2(r)) < D, r \gg 1$. Let $\xi_j := (x, \gamma'_j(0)) \in SM$. Then

$$\frac{Jac_{U(\xi_1)} G^r}{Jac_{U(\xi_2)} G^r} < C.$$

Proof: Let $\xi_j(s) := (\gamma_j(s), \gamma_j'(s)) \in SM$, where $0 \le s \le r$. Then

$$\ln Jac_{U(\xi_j)} G^r \asymp \int_0^r \mathcal{H}(\xi_j(s)) ds.$$

Fact:

$$\mathrm{Dist}_{SM}(\xi_1(s), \xi_2(s)) \le Ce^{\beta(s-r)}.$$

Lemma 10 follows from Hölder continuity of $\mathcal{H}$. Q.E.D.

Consider a *primitive* closed geodesic γ on X of length $l(\gamma)$. It corresponds to a conjugacy class $[\omega(\gamma)] \in \Gamma = \pi_1(X)$.

D - diameter of the Poincare fundamental domain for Γ in M. Choose generators $\{a_j\}$ for Γ, and let $w(\gamma)$ be the corresponding word in a_j-s of word length $l(w(\gamma)) \ge l(\gamma)/D$.

Theorem (Preissman) $\Rightarrow$ all $l(w(\gamma))$ cyclic shifts of of $w(\gamma)$ are distinct elements of Γ.

Key step: Group $\omega \in \Gamma$ into conjugacy classes $[\omega(\gamma)]$ parametrized by closed geodesics γ on X.

Lemma 11. to each primitive γ corresponds *at least* $l(\gamma)/D$ elements $\omega_i(\gamma) \in \Gamma$ such that the conclusion of Lemma 10 applies to $\gamma_1 = [x, \omega_i y]$ (geodesic segment from x to $\omega_i y$) and $\gamma_2 = \gamma$.

Proof: Can choose $z_i \in \gamma$ so that

$$\mathrm{dist}_M(x, z_i) < D, \quad \mathrm{dist}_M(\omega_i y, \omega_i z_i) < D,$$

and $[z_i, \omega_i z_i] = \gamma$. Then apply Lemma 10 twice. Q.E.D.

Step 3: For a closed geodesic γ on X, let $\xi(s,\gamma) := (\gamma(s), \gamma'(s)) \in SM$.

Corollary 9, Lemmas 10 and 11 $\Rightarrow$

Corollary 12.

$$S_{x,y}(T) \;\geq\; \frac{C}{D}\times$$

$$\sum_{\gamma : l(\gamma) < T} l(\gamma) \exp\left(\frac{-1}{2} \int_0^{l(\gamma)} \mathcal{H}(\xi(s,\gamma)) ds \right).$$

By results of Parry (1986) and Parry-Pollicott (1990), the sum above is asymptotic to

$$Ce^{P\left(-\frac{\mathcal{H}}{2}\right)\cdot T}, \qquad C > 0.$$

This finishes the proof of Theorem 7. Q.E.D.