ON THE DISTRIBUTION OF PERTURBATIONS OF PROPAGATED SCHRÖDINGER EIGENFUNCTIONS

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Abstract. Let (M, g_0) be a compact Riemmanian manifold of dimension n. Let $P_0(h) := -h^2\Delta_g + V$ be the semiclassical Schrödinger operator for $h \in (0, h_0]$, and let E be a regular value of its principal symbol $p_0(x,\xi) = |\xi|^2_{g_0(x)} + V(x)$. Write φ_h for an L^2-normalized eigenfunction of $P(h)$, $P_0(h)\varphi_h = E(h)\varphi_h$ and $E(h) \in [E - o(1), E + o(1)]$. Consider a smooth family of perturbations g_u of g_0 with u in the ball $\mathcal{B}^k(\varepsilon) \subset \mathbb{R}^k$ of radius $\varepsilon > 0$. For $P_u(h) := -h^2\Delta_{g_u} + V$ and small $|t|$, we define the propagated perturbed eigenfunctions

$$\varphi_h^{(u)} := e^{-\frac{i}{h}tP_u(h)}\varphi_h.$$

We study the distribution of the real part of the perturbed eigenfunctions regarded as random variables

$$\Re\left(\varphi_h^{(\cdot)}(x)\right) : \mathcal{B}^k(\varepsilon) \to \mathbb{R} \qquad \text{for } x \in M.$$

In particular, when (M, g) is ergodic, we compute the $h \to 0^+$ asymptotics of the variance $\mathrm{Var}\left[\Re\left(\varphi_h^{(\cdot)}(x)\right)\right]$ and show that all odd moments vanish as $h \to 0^+$.

1. Introduction

Let (M, g_0) be a compact Riemmanian manifold of dimension n with Laplace operator $\Delta_{g_0} = \delta_{g_0} d : C^\infty(M) \to C^\infty(M)$ and let $V \in C^\infty(M)$ denote a smooth potential over M. For $h \in (0, h_0]$, consider the Schrödinger operator

$$P_0(h) := -h^2\Delta_{g_0} + V, \tag{1}$$

and let E be a regular value of its principal symbol $p_0(x,\xi) := |\xi|^2_{g_0(x)} + V(x)$.

Write φ_h for an L^2-normalized eigenfunction of $P(h)$ belonging to an energy shell centered at E; that is, $P_0(h)\varphi_h = E(h)\varphi_h$ and $E(h) \in [E - o(1), E + o(1)]$.

Consider a smooth family of perturbations g_u of the reference metric g_0 with u in the ball $\mathcal{B}^k(\varepsilon) \subset \mathbb{R}^k$ of radius $\varepsilon > 0$. The number of parameters $k \geq n$ is chosen sufficiently large (but finite) so that the admissibility condition on the perturbation g_u in Definition 1 is satisfied. We introduce the associated perturbed Schrödinger

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operators

$$P_u(h) := -h^2\Delta_{g_u} + V, \tag{2}$$

with principal symbol

$$p_u : T^*M \to T^*M, \qquad p_u(x,\xi) := |\xi|^2_{g_u} + V(x). \tag{3}$$

Let H_{p_u} be the Hamiltonian vector field on T^*M induced by p_u, and write $G_u^s : T^*M \to T^*M$ for the bicharacteristic flow associated to H_{p_u} at time s.

We define the propagated perturbed eigenfunctions

$$\varphi_h^{(u)} := e^{-\frac{i}{h}tP_u(h)}\varphi_h. \tag{4}$$

These are the solutions at time t of the Schrödinger equation

$$\begin{cases} \left(ih\frac{\partial}{\partial s} - P_u(h)\right)\Phi_h^{(u)}(s) = 0, \\ \Phi_h^{(u)}(0) = \varphi_h. \end{cases}$$

It follows that $\Phi_h^{(u)}(t) = \varphi_h^{(u)}$.

The aim of this paper is to study the $h \to 0$ asymptotics of the distribution of $\varphi_h^{(u)}$, where the latter are regarded as random variables in $u \in \mathcal{B}^k(\varepsilon)$. Specifically, we compute the variance and all odd moments in the semiclassical limit $h \to 0^+$.

To state our results, we need to define the admissibility condition on the metric perturbations.

Definition 1 (Admissibility condition).
Let g_u with $u \in \mathcal{B}^k(\varepsilon)$ be a metric perturbation of a reference metric g_0. We say that g_u is **admissible** at $x \in M$ if

A) There exists a constant $c > 0$ and an n-tuple of coordinates of u, $u' = (u_1, \dots, u_n)$, for which the Hessian matrices
$$d_{u'}d_\xi(p_u(x,\xi))\big|_{u=0}$$
are invertible for all $\xi \in T_x^*M$ with $(x,\xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$.

B) There exists a pointwise conformal direction given by a variable $u_\alpha \in (-\epsilon,\epsilon)$ in which g_0 is non-trivially perturbed at x. That is, there exists a small neighborhood $\mathcal{W}$ of x and $a \in C^\infty(\mathcal{W})$ with $a(y) \neq 0$ for all $y \in \mathcal{W}$ so that
$$\delta_{u_\alpha} g_u(y) := \partial_{u_\alpha} g_u(y)|_{u=0} = a(y)\, g_0(y) \qquad \forall y \in \mathcal{W}.$$

We show in Section 5 that the admissibility condition in Definition 1 is satisfied by a large class of metric perturbations and we also give a geometric interpretation of the admissibility condition.

The perturbed eigenfunctions $\varphi_h^{(u)}$ are regarded here as random variables in the deformation parameters $u \in \mathcal{B}^k(\varepsilon)$ and so we endow the ball $\mathcal{B}^k(\varepsilon)$ with the probability measure $|\mathcal{B}^k(\varepsilon)|^{-1}du$ where $|\mathcal{B}^k(\varepsilon)|$ denotes the volume of the ball $\mathcal{B}^k(\varepsilon)$ in $\mathbb{R}^k$. We view the real part of the perturbed eigenfunctions $\varphi_h^{(u)}$ defined in (4) as random variables

$$\Re\left(\varphi_h^{(\cdot)}(x)\right) : \mathcal{B}^k(\varepsilon) \to \mathbb{R}$$

depending on the spatial parameters $x \in M$.

Since one can study the distribution of a random variable such as $\Re\left(\varphi_h^{(\cdot)}(x)\right)$ by understanding its moments, in this paper we study the asymptotics of the variance $Var\left[\Re\left(\varphi_h^{(\cdot)}(x)\right)\right]$ and of the odd moments

$$\mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right]^p$$

in the semiclassical limit $\hbar \to 0^+$.

Our first result holds for general Riemannian manifolds (M, g_0).

Theorem 1. *Let (M, g_0) be a compact Riemannian manifold of dimension n and let E be a regular value of p_0. Suppose g_u is a perturbation of g_0 with $u \in \mathcal{B}^k(\varepsilon) \subset \mathbb{R}^k$ that is admissible at every $x \in M$. Then, for $\varepsilon > 0$ and $|t| > 0$ be sufficiently small depending on (M, g_0) there is $h_0(t,\varepsilon) > 0$ such that for $h \in (0, h_0(t,\varepsilon)]$,*

(1) There exist positive constants $C_1 = C_1(M, g_0)$ and $C_2 = C_2(M, g_0)$ with

$$C_1 \leq Var\left[\Re\left(\varphi_h^{(\cdot)}(x)\right)\right] \leq C_2.$$

(2) For $p \in \mathbb{Z}^+$ odd,

$$\mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right]^p = \mathcal{O}(h^\infty).$$

Moreover, these estimates are locally uniform in $x \in (V^{-1}(E))^c$.

If the metric perturbation g_u is admissible, there exist $c > 0$ and an n-tuple of u-coordinates denoted by $u' = (u_1, \ldots, u_n) \in \mathcal{B}^n(\varepsilon)$ for which $|d_{u'} d_\xi p_{(u',u'')}(x,\eta)| \neq 0$ at $u = 0$ provided $(x,\eta) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$. Using this, we show via an Implicit Function Theorem argument that for the relevant generating function $S(t,u,y,\eta)$ in (12) and for points $(u', \tau; y, \eta) \in \Gamma_{x,u''}$ (29) where the Lagrangian

$$\Gamma_{x,u''} = \left\{(u', d_{u'}S(t,u,\eta;x)), d_\eta S(t,u,\eta;x), \eta);\ (d_\eta S(t,u,\eta;x), \eta) \in supp\, \chi_E^{(0)}\right\} \subset T^*\mathcal{B}^n(\varepsilon) \times T^*M,$$

one can locally parametrize u' as a smooth function of $(y,\eta) \in p_0^{-1}(E)$, $u' = u'(y,\eta)$. We write $u'' \in \mathcal{B}^{k-n}(\varepsilon)$ for the omitted parameters and the dependence of $u'(y,\eta)$ on (u'', x) as parameters is understood. Furthermore, without loss of generality, we assume that the coordinates of u are ordered so that $u = (u', u'')$.

In the case where the manifold (M, g_0) has an ergodic geodesic flow $G^t : S^*M \to S^*M$, we get asymptotic results for the variance. In the following, we say that a sequence of L^2-normalized eigenfunctions (φ_h) of $P_0(h)$ with $P_0(h)\varphi_h = E(h)\varphi_h$ and $E(h) = E + o(1)$ is quantum ergodic (QE) if for any $a \in S^{0,0}_{cl}(T^*M \times [0, h_0))$,

$$\langle Op_h(a)\varphi_h, \varphi_h\rangle \sim_{h\to 0^+} \int_{p_0^{-1}(E)} a(x,\xi)d\omega_E(x,\xi)$$

whre, $d\omega_E$ is Liouville measure on $p_0^{-1}(E)$.

Theorem 2. *Let (M, g_0) be a compact Riemannian manifold of dimension n and let E be a regular value of p_0. Assume the geodesic flow on $p_0^{-1}(E)$ is ergodic and that $\{\varphi_h\}_{h\in(0,h_0]}$ is a quantum ergodic sequence of L^2-normalized eigenfunctions of $P_0(h)$. Suppose g_u with $u \in \mathcal{B}^k(\varepsilon)$ is a perturbation of g_0 that is admissible at $x \in (V^{-1}(E))^c$ and that $|t| > 0$ and $\varepsilon > 0$ are sufficiently small.*

(1) Then,

$$\lim_{h\to 0^+} Var\left[\Re(\varphi_h^{(\cdot)}(x))\right] = \frac{1}{|\mathcal{B}^k(\varepsilon)|}\int_{B^{k-n}(\varepsilon)} \beta_x^k(u'')\, du''$$

where $\beta_x^k : B^{k-n}(\varepsilon) \to \mathbb{R}$ is defined by

$$\beta_x^k(u'') := \frac{1}{|t|^n|p_0^{-1}(E)|}\int_{p_0^{-1}(E)} \frac{|d_x\pi G^t_{(u'(y,\eta),u'')}(x,\eta)|}{|\, d_{u'}d_\xi\; p_{(u'(y,\eta),u'')}(x,\eta)|}\, d\omega_E(y,\eta). \tag{5}$$

(2) For $p \in \mathbb{Z}^+$ odd,

$$\lim_{h\to 0^+}\; \mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right]^p = 0.$$

1.1. **Motivation.** We proceed to describe two ideas that motivate our work. We first explain how the underlying ideas in our approach are motivated by the random wave conjecture. We then relate our results to the physics notion of Loschmidt echo.

Random wave conjecture. In 1971 M. Berry conjectured that the real and imaginary parts of the eigenfunctions φ_h in the chaotic case resemble random waves, [1]. It is also believed that the eigenfunctions φ_h of quantum mixing systems behave locally as independent gaussian variables as $h \to 0$; see for example the discussion in [9] and references therein. One of the common issues is to define a probability model where the eigenfunctions can be thought of as random variables. This is the role we give to the perturbations $\varphi_h^{(u)}$.

Loschmidt echo. A natural way of measuring the noise affecting a given system is the Loschmidt echo. The idea behind this concept is to measure the sensitivity of quantum evolution to perturbations, by propagating forward an initial state ψ using the unperturbed hamiltonian p_0 and after time t propagate it back via the perturbed one p_u. Thus, the objects of interest in this case are the states $e^{\frac{it}{h}P_u(h)}e^{-\frac{it}{h}P_0(h)}\psi$ and the *Loschmidt Echo*, $M_{LE}(t)$, is defined to be the return probablility to the initial state:

$$M_{LE}(t) = \left|\langle e^{-\frac{it}{h}P_u(h)}e^{\frac{it}{h}P_0(h)}\psi, \psi\rangle\right|^2.$$

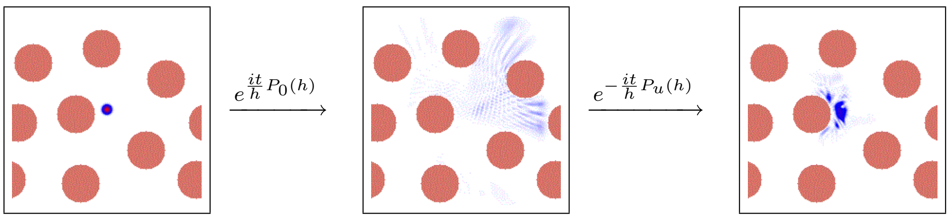

Illustration of the state of particle initially placed in the center of a square billiard with an irregular array of 10 circular scatterers with initial momentum pointing to the left [3].

We are interested in the case when the initial state ψ is an eigenfunction, $\psi = \varphi_h$. In this simpler case $M_{LE}(t)$ is called the *survival probability* [16] and we have

$$e^{\frac{it}{h}P_u(h)}e^{-\frac{it}{h}P_0(h)}\varphi_h = e^{-\frac{itE(h)}{h}}\varphi_h^{(u)}.$$

To be precise, for an initial state φ_h the Loschmidt Echo is simply

$$M_{LE}(t) = \left|\langle e^{-\frac{it}{h}P_u(h)}e^{\frac{it}{h}P_0(h)}\varphi_h, \varphi_h\rangle\right|^2 = \left|\langle \varphi_h^{(u)}, \varphi_h\rangle\right|^2.$$

As the definition shows, the fidelity $M_{LE}(t)$ can be interpreted as the decaying overlap between the evolution $\varphi_h^{(u)}$ and the unperturbed evolution φ_h, [11, 10, 13].

In recent work [8], Eswarathasan and Toth have proved related results for *magnetic* deformations of the Hamiltonian $p_0(x,\xi) = |\xi|^2_{g(x)} + V(x)$. We extend their results here to large families of metric deformations. In additon, we characterize the asymptotic results in terms of variance and show that all odd moments are negligible up to large order depending on the dimension n and the number of parameters, k. Although we do not have a rigorous argument at the moment, we hope that by further developing the methods of the present paper, we will be able to compute the higher even moments $\lim_{h\to 0^+} \mathbb{E}[\Re(\varphi_h^{(\cdot)}(x))]^{2p}$ for $p \geq 2$, and compare them with the Gaussian prediction of the random wave model. We plan to return to this question elsewhere.

1.2. **Outline of the paper.**

In Section 2 we introduce the background material and notation from semiclassical analysis that we shall use to prove our results. We first show that the perturbations are semiclassically localized in $p_0^{-1}(E)$ and then explain how to microlocally cut off the propagators $e^{-\frac{it}{h}P_u(h)}$ to obtain a localized approximation of $\varphi_h^{(u)}$. The material here is standard [17] but we have included it here for the benefit of the reader.

In Section 3, we study the odd moments of $\Re(\varphi_h^{(\cdot)}(x))$. Provided the metric perturbation satisfies part (B) of the admissibility condition at $x_* \in M$, we prove in Proposition 4 that for $\varepsilon > 0$, $|t| > 0$ small, and $\ell, q \in \mathbb{Z}^+$ there exists $\tau_\varepsilon > 0$ so that for $h \in (0, h_0(t,\varepsilon)]$,

$$\int_{\mathcal{B}^k(\varepsilon)} \left(\varphi_h^{(u)}(x)\right)^\ell \left|\varphi_h^{(u)}(x)\right|^{2q} du = \mathcal{O}(h^\infty),$$

uniformly in $x \in B(x_*, \tau_\varepsilon) \cap (V^{-1}(E))^c$. Using Proposition 4 and the binomial expansion for $(\varphi + \bar{\varphi})^p = (2\Re\varphi)^p$, we prove that for $p \in \mathbb{Z}^+$ odd,

$$\mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right]^p = \mathcal{O}(h^\infty), \tag{6}$$

uniformly in $x \in B(x_*, \tau_\varepsilon) \cap (V^{-1}(E))^c$.

In Section 4 we study the variance of $\Re(\varphi_h^{(\cdot)}(x))$. Provided the perturbation is admissible at $x \in (V^{-1}(E))^c$, the case of $p = 1$ in (6) shows that our variables are semiclassically centered with

$$\mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right] = \mathcal{O}(h^\infty).$$

Therefore,

$$Var\left[\Re(\varphi_h^{(\cdot)}(x))\right] = \frac{1}{|B^k(\varepsilon)|}\int_{\mathcal{B}^k(\varepsilon)} |\varphi_h^{(u)}(x)|^2\, du + \mathcal{O}(h^\infty). \tag{7}$$

It follows that studying the variance is equivalent to understanding the behavior of the right hand side in the previous equality. Using Proposition 5, we compute the asymptotics of the RHS in (7) and prove Theorem 1 and Theorem 2.

In Section 5 we show that there are always exist large families of admissible perturbations and that the notion of admissibility is related to having sufficiently many volume preserving directions in which the metric tensor g_u is perturbed.

Remark 1. We note here that there is an easy consequence of Theorem 1 that concerns restriction bounds of $\varphi_h^{(u)}$ to submanifolds $H \subset M$. Indeed, since the bounds in Theorem 1 are locally uniform in $x \in M$, by covering the submanifold $H \subset M$ with finitely-many small balls, integrating over H and applying Fubini, one gets that for $h \in (0, h_0]$, there are constants $C_j = C_j(H, h_0) > 0; j = 1, 2$, with

$$C_1 \leq \int_{\mathcal{B}^k(\varepsilon)} \int_H |\varphi_h^{(u)}(s)|^2\, d\sigma_H(s) du \leq C_2.$$

By the Tschebyshev inequality, it then follows that for *any* sequence $\omega(h) = o(1)$ as $h \to 0^+$, there is a measurable $D(h) \subset B^k(\epsilon)$ with $\lim_{h\to 0^+} \frac{|D(h)|}{|B^k(\epsilon)|} = 1$ such that for $u \in D(h)$,

$$\int_H |\varphi_h^{(u)}(s)|^2 d\sigma_H(s) = \mathcal{O}(|\omega(h)|^{-1}).$$

Therefore, the restriction bounds for most perturbed eigenfunctions are much smaller than the universal bounds for $\int_H |\varphi_h^{(0)}(s)|^2 d\sigma_H(s)$ in [2, Theorem 3] and tend to be consistent with the ergodic case [5, 15].

2. Background and Notation

In this section we introduce some background material on eigenfunction localization and semiclassically cut off propagators. Most of this is standard in semiclassical analysis, but we include it for the benefit of the reader. We refer to [17] for further details. Let M be a compact Riemannian manifold of dimension n. We work with the class of semiclassical symbols

$$S_{cl}^{m,k}(T^*M) := \Big\{ a \in C^\infty(T^*M \times (0, h_0]) : \;\; a(x,\xi;h) \sim_{h\to 0^+} h^{-m} \sum_{j=1}^{\infty} a_j(x,\xi)\, h^j \\ \text{with} \;\; |\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \leq C_{\alpha,\beta}\,(1+|\xi|^2)^{\frac{k-|\beta|}{2}} \Big\}.$$

For $a \in S_{cl}^{m,k}(T^*M)$, consider the Schwartz kernel in $M \times M$ locally defined by

$$Op_h(a)(x,y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi;h)\, d\xi.$$

The corresponding space of pseudodifferential operators is defined to be

$$\Psi_{cl}^{m,k}(M) := \{ Op_h(a) : \;\; a \in S_{cl}^{m,k}(T^*M) \}.$$

Let N be another compact n-dimensional Riemannian manifold. We also consider the class of Fourier integral operators $I_{cl}^{m,k}(M \times N, \Gamma)$ with Schwartz kernels locally defined in the form

$$F_h(x,y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,y,\xi)} a(x,y,\xi;h)\, d\xi$$

for $a \in C_0^\infty(U \times V \times \mathbb{R}^n \times (0,h_0])$ with $a(x,y,\xi;h) \sim_{h\to 0^+} h^{-m} \sum_{j=1}^\infty a_j(x,y,\xi)\, h^j$ where $U, V \subset \mathbb{R}^n$ are local coordinate charts. Here ϕ denotes a non-degenerate phase function in the sense of Hörmander [4, Def (2.3.10)] and Γ is an immersed Lagrangian submanifold

$$\Gamma = \{ (x, d_x\phi, \; y, -d_y\phi) : \;\; d_\xi \phi(x,y,\xi) = 0 \} \subset T^*M \times T^*N.$$

2.1. **Eigenfunction localization.** For E a regular value of p_0 and $u \in \mathcal{B}^k(\varepsilon)$ we introduce the cut-off functions on T^*M

$$\chi_E^{(u)}(x,\xi) = \chi(p_0(G_u^t(x,\xi)) - E), \tag{8}$$

where $\chi \in C_0^\infty(\mathbb{R};[0,1])$ denotes a standard cut-off function equal to 1 near the origin. Observe that since $\chi_E^{(u)}(x,\xi) = \chi(p_0(x,\xi) - E + \mathcal{O}(|u|))$, the support of $\chi_E^{(u)}$ remains localized near the hypersurface $p_0^{-1}(E)$ for all $u \in \mathcal{B}^k(\varepsilon)$.

Note that $\varphi_h^{(u)}$ is a normalized eigenfunction of the operator

$$Q_u(h) := e^{-\frac{i}{h}tP_u(h)} P_0(h) e^{\frac{i}{h}tP_u(h)} \in \Psi_{cl}^{0,2}(M)$$

with eigenvalue $E(h)$. By Egorov's Theorem $Q_u(h) = Op_h(p_0 \circ G_u^t) + \mathcal{O}_{L^2 \to L^2}(h)$, and since $E(h) \in [E - o(1), E + o(1)]$, we obtain $(Q_u(h) - E)\varphi_h^{(u)} = o(1)$. Using that $Q_u(h)$ is h-elliptic off $(p_0 \circ G_u^t)^{-1}(E)$, a parametrix construction [17, Thm. 6.4] gives $\|\varphi_h^{(u)} - Op_h(\chi_E^{(u)})\varphi_h^{(u)}\|_{L^2} = \mathcal{O}(h^\infty)$ and therefore $WF_h(\varphi_h^{(u)}) \subset (p_0 \circ G_u^t)^{-1}(E)$. Since $(p_0 \circ G_u^t)^{-1}(E) \subset p_0^{-1}(E - c|u|, E + c|u|))$ for some $c > 0$ and $u \in \mathcal{B}^k(\varepsilon)$, we obtain $WF_h(\varphi_h^{(u)}) \subset p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$. By a Sobolev lemma argument one can also prove $\|\varphi_h^{(u)} - Op_h(\chi_E^{(u)})\varphi_h^{(u)}\|_{C^k} = \mathcal{O}_{C^k}(h^\infty)$. It follows that

$$\varphi_h^{(u)} = Op_h(\chi_E^{(u)}) \circ e^{-\frac{it}{h}P_u(h)} \circ Op_h(\chi_E^{(0)})\, \varphi_h + \mathcal{O}_{C^k}(h^\infty). \tag{9}$$

2.2. **Semiclassically cut off propagators.** Motivated by the approximation (9), for $h \in (0, h_0]$, $u \in \mathcal{B}^k(\varepsilon)$ and $|t|$ small, we define the semiclassically cut off Fourier integral operators $W_u(h) \in I_{cl}^{0,-\infty}(M \times M, \Gamma_u)$,

$$W_u(h) := Op_h(\chi_E^{(u)}) \circ e^{-\frac{it}{h}P_u(h)} \circ Op_h(\chi_E^{(0)}), \tag{10}$$

associated with the immersed Lagrangian,

$$\Gamma_u = \Big\{(x, \xi; y, \eta) :\ (x, \xi) = G_u^t(y, \eta) \in supp\, \chi_E^{(u)} \quad \text{and} \ \ (y, \eta) \in supp\, \chi_E^{(0)}\Big\} \\ \subset T^*M \times T^*M.$$

We note that since G_u^s is a symplectomorphism there exists a local generating function $S(s, u, \xi; x)$ with $(x, \partial_x S(s, u, \eta; x)) = G_u^t(\partial_\eta S(s, u, \eta; x, \eta), \eta)$ for s close to t. It follows that

$$\Gamma_u = \Big\{\big(x, d_x S(t, u, \eta; x);\ d_\eta S(t, u, \eta; x), \eta\big) \in supp\, \chi_E^{(u)} \oplus supp\, \chi_E^{(0)}\Big\} \\ \subset T^*M \times T^*M. \tag{11}$$

The generating function $S(s, u, \eta; x)$ solves the Hamilton-Jacobi initial value problem

$$\begin{cases} \partial_s S(s, u, \eta; x) + p_u(x, \partial_x S(s, u, \eta; x)) = 0, \\ S(0, u, \eta; x) = \langle x, \eta \rangle, \end{cases}$$

and therefore, a Taylor expansion in s around $s = 0$ gives

$$S(s, u, \eta; x) = \langle x, \eta \rangle - s\, p_u(x, \eta) + \mathcal{O}(s^2). \tag{12}$$

Given local coordinate charts $U, V \subset \mathbb{R}^n$ consider the local phase function $\phi \in C^\infty(V \times \mathcal{B}^k(\varepsilon) \times \mathbb{R}^n)$,

$$\phi(y, u, \xi; x) := S(t, u, \xi; x) - \langle y, \xi \rangle, \tag{13}$$

for $u \in U \times \mathcal{B}^k(\varepsilon)$. The Schwartz kernel of $W_u(h)$ is given by

$$W_u(h)(x, y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(y, u, \xi; x)} a(u, y, \xi; x, h)\, d\xi + K_x(y, u), \tag{14}$$

where $|\partial_x^\alpha \partial_y^\beta K_x(y, u)| = \mathcal{O}_{\alpha,\beta}(h^\infty)$ uniformly in $(x, y, u) \in U \times V \times \mathcal{B}^k(\varepsilon)$ for $\varepsilon > 0$ small, for $U, V \subset \mathbb{R}^n$ coordinate charts.
The amplitude $a(u, y, \xi; x, h) \sim \sum_{j=0}^\infty a_j(u, y, \xi; x) h^j$ with

$$a_j(u, \cdot, \cdot; \cdot) \in C^\infty(B^k(\varepsilon), C_0^\infty(V \times \mathbb{R}^n \times U)).$$

Since the support of $\chi_E^{(u)}$ remains localized near the hypersurface $p_0^{-1}(E)$ for all $u \in \mathcal{B}^k(\varepsilon)$, there exists $c > 0$ with

$$supp\ (a(u, y, \cdot\,; x, h)) \subset \{\xi \in T_x^*M : (x, \xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)\}.$$

3. Odd moments

The purpose of this section is to show that provided the metric is deformed in a pointwise conformal direction, its odd moments are negligible for general geodesic flows. Throughout this section assume (M, g_0) is a compact Riemannian manifold and E is a regular value of p_0. We prove

Proposition 3. *Let g_u with $u \in \mathcal{B}^k(\varepsilon)$ be a perturbation of g_0 that satisfies part* (B) *of the admissibility condition at $x_* \in M$. Cosider $p \in \mathbb{Z}^+$ odd. For $\varepsilon > 0$ and $|t|$ small, there exists $\tau_\varepsilon > 0$ so that*

$$\mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right]^p = \mathcal{O}(h^\infty), \qquad \textit{as } h \to 0^+, \tag{15}$$

uniformly in $x \in B(x_*, \tau_\varepsilon) \cap (V^{-1}(E))^c$.

Proof. Observe that

$$\mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right]^p = \frac{1}{|\mathcal{B}^k(\varepsilon)|}\int_{\mathcal{B}^k(\varepsilon)} \left(\Re(\varphi_h^{(u)}(x))\right)^p\, du,$$

and for any complex φ the binomial expansion of $(\varphi + \bar{\varphi})^p = (2\Re\varphi)^p$ for p odd gives

$$(\Re\varphi)^p = \frac{1}{2^p}\sum_{0\le j<\frac{p}{2}} \binom{p}{j}\varphi^{p-2j}\,|\varphi|^{2j} + \frac{1}{2^p}\sum_{\frac{p}{2}<j\le p}\binom{p}{j}\bar{\varphi}^{\,2j-p}\,|\varphi|^{2(p-j)}. \tag{16}$$

Therefore, to prove Proposition 3, it suffices to show that there exists $\tau_\varepsilon > 0$ making

$$\int_{\mathcal{B}^k(\varepsilon)} \left(\varphi_h^{(u)}(x)\right)^\ell \left|\varphi_h^{(u)}(x)\right|^{2q} du = \mathcal{O}(h^\infty) \qquad \text{for} \quad 1 \le \ell \le p,\ 2q \le p, \tag{17}$$

locally uniformly in $x \in B(x_*, \tau_\varepsilon) \cap (V^{-1}(E))^c$ as $h \to 0^+$.

Since the proof of (17) is somewhat technical we prove it as a separate Lemma. Combining (17) with the binomial expansion (16) completes the proof. □

We have reduced the proof of Proposition 3 to establishing the following

Lemma 4. *Let g_u with $u \in \mathcal{B}^k(\varepsilon)$ be a perturbation of g_0 that satisfies part* (B) *of the admissibility condition at $x_* \in M$. Suppose $\ell, q \in \mathbb{Z}^+$. Then, for all $\varepsilon > 0$ and $|t|$ small, there exists $\tau_\varepsilon > 0$ making*

$$\int_{\mathcal{B}^k(\varepsilon)} \left(\varphi_h^{(u)}(x)\right)^\ell \left|\varphi_h^{(u)}(x)\right|^{2q} du = \mathcal{O}(h^\infty), \qquad \textit{as } h \to 0^+,$$

uniformly in $x \in B(x_*, \tau_\varepsilon) \cap (V^{-1}(E))^c$.

Proof. We identify $\oplus^{\ell+2q}M$ with $(\oplus^{\ell}M)\oplus(\oplus^{q}M)\oplus(\oplus^{q}M)$ and write $(\tilde y,\tilde z,\tilde z'):=(y^{(1)},\dots,y^{(\ell)},z^{(1)},\dots,z^{(q)},z'^{(1)},\dots,z'^{(q)})\in\oplus^{\ell+2q}M$.

By assumption, there exists $0<\tau_\varepsilon<inj(M)$ and $a\in C^\infty(M)$ so that $\delta_{u_\alpha}g_u(x)=a(x)\,g_0(x)$ with $a(x)\neq 0$ for all $x\in B(x_*,\tau_\varepsilon)$.

Since from (9), $\varphi_h^{(u)}(x)=[W_u(h)\varphi_h](x)+\mathcal{O}(h^\infty)$, writing $W_u(x,y)$ for the kernel of W_u we get

$$\int_{\mathcal{B}^k(\varepsilon)}\left(\varphi_h^{(u)}(x)\right)^\ell\left|\varphi_h^{(u)}(x)\right|^{2q}\,du+\mathcal{O}(h^\infty)= \tag{18}$$
$$=\int\limits_{\mathcal{B}^k(\varepsilon)}\left([W_u(h)\varphi_h](x)\right)^\ell\left|[W_u(h)\varphi_h](x)\right|^{2q}\,du$$
$$=\int\limits_{\mathcal{B}^k(\varepsilon)}\int\limits_{M^{\ell+2q}}\prod_{\substack{1\le i\le\ell\\1\le j\le q}}W_u(x,y^{(i)})W_u(x,z^{(j)})\overline{W_u(x,z'^{(j)})}\varphi_h(y^{(i)})\varphi_h(z^{(j)})\overline{\varphi_h(z'^{(j)})}d\tilde y d\tilde z d\tilde z' du$$
$$=\int\limits_{\mathcal{B}^k(\varepsilon)}\int\limits_{M^{\ell+2q}}B_u^{[\ell,q]}(\tilde y,\tilde z,\tilde z';x,h)\,\varphi_h(y^{(i)})\,\varphi_h(z^{(j)})\,\overline{\varphi_h(z'^{(j)})}\,d\tilde y\,d\tilde z\,d\tilde z'\,du,$$

where $B_u^{[\ell,q]}\in C^\infty(\oplus^{\ell+2q}M\times M\times[0,h_0))$ is defined by

$$B_u^{[\ell,q]}(\tilde y,\tilde z,\tilde z';x,h):=\prod_{\substack{1\le i\le\ell\\1\le j\le q}}W_u(x,y^{(i)})W_u(x,z^{(j)})\overline{W_u(x,z'^{(j)})}.$$

From (14),

$$B_u^{[\ell,q]}(\tilde y,\tilde z,\tilde z';x,h)=$$
$$=\frac{1}{(2\pi h)^{n(q+\frac{\ell}{2})}}\int\limits_{\mathcal{B}^{n(\ell+q)}(\varepsilon)}\int\limits_{\mathbb{R}^{nq}}\int\limits_{\mathbb{R}^{nq}}\int\limits_{\mathbb{R}^{n\ell}}e^{\frac{i}{h}\Phi^{[\ell,q]}(\tilde y,\tilde z,\tilde z',u,\tilde\xi,\tilde\eta,\tilde\eta';x)}\;c^{[\ell,q]}(u,\tilde y,\tilde z,\tilde z',\tilde\xi,\tilde\eta,\tilde\eta';x,h)$$
$$\times\,d\tilde\xi\,d\tilde\eta\,d\tilde\eta'\,du'$$
$$+\,K_x(\tilde y,\tilde z,\tilde z',u), \tag{19}$$

for $\Phi^{[\ell,q]}$, $c^{[\ell,q]}$ and K_x as follows:

(i) The phase function $\Phi^{[\ell,q]}$ is defined by

$$\Phi^{[\ell,q]}(\tilde y,\tilde z,\tilde z',u,\tilde\xi,\tilde\eta,\tilde\eta';x):=$$
$$=\sum_{j=1}^{\ell}\phi\left(y^{(j)},u,\xi^{(j)};x\right)+\sum_{j=1}^{q}\phi\left(z^{(j)},u,\eta^{(j)};x\right)-\phi\left(z'^{(j)},u,\eta'^{(j)};x\right),$$

where ϕ is as in (13).

(ii) The amplitude $c^{[\ell,q]}$ satisfies

$$c^{[\ell,q]}(u,\tilde{y},\tilde{z},\tilde{z}',\tilde{\xi},\tilde{\eta},\tilde{\eta}'\,;x,h) \sim_{h\to 0^+} \sum_{j=0}^{\infty} c_j^{[\ell,q]}(u,\tilde{y},\tilde{z},\tilde{z}',\tilde{\xi},\tilde{\eta},\tilde{\eta}'\,;x)h^j$$

with $c_j^{[\ell,q]}(u,\cdot,\cdot\,;\cdot) \in C^\infty\left(B^k(\varepsilon), C_0^\infty(V\times\mathbb{R}^{n(\ell+2q)}\times U)\right)$ for $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^{n(2q+\ell)}$ local coordinate charts. Moreover,

$$\begin{aligned}
&supp\,(c^{[\ell,q]}(u,\tilde{y},\tilde{z},\tilde{z}',\cdot\,;x,h)) \subset \\
&\left\{(\tilde{\xi},\tilde{\eta},\tilde{\eta}'):\quad (x,\xi^{(i)}),(x,\eta^{(j)}),(x,{\eta'}^{(j)})\in p_0^{-1}(E-c\varepsilon,E+c\varepsilon),\quad i\le\ell,\ j\le q\right\}\\
&\subset\mathbb{R}^{n(\ell+2q)}.
\end{aligned}\tag{20}$$

(iii) The residual operator K_x satisfies

$$|\partial_x^\alpha\,\partial_{(\tilde{y},\tilde{z},\tilde{z}')}^\beta\,K_x(\tilde{y},\tilde{z},\tilde{z}',u)| = \mathcal{O}_{\alpha,\beta}(h^\infty)$$

locally uniformly in $(\tilde{y},\tilde{z},\tilde{z}',u)\in\oplus^{\ell+2q}M\times\mathcal{B}^k(\varepsilon)$.

Claim. For $\varepsilon>0$ and $|t|>0$ sufficiently small, there exists $C=C(t,\varepsilon,E,g_0)>0$ such that for $(\tilde{\xi},\tilde{\eta},\tilde{\eta}')\in supp\,(c^{[\ell,q]}(u,\tilde{y},\tilde{z},\tilde{z}',\cdot\,;x,h))$,

$$\left|\delta_{u_\alpha}\Phi^{[\ell,q]}(\tilde{y},\tilde{z},\tilde{z}',u,\tilde{\xi},\tilde{\xi}',\tilde{\eta},\tilde{\eta}';x)\right| \ge C>0. \tag{21}$$

Moreover, this bound holds locally uniformly in $(\tilde{y},\tilde{z},\tilde{z}')\in\oplus^{\ell+2q}M$, and $u\in B^k(\varepsilon)$, and $x\in B(x_*,\tau_\varepsilon)\cap(V^{-1}(E))^c$.

To prove this claim we first observe that in normal coordinates centered at x_*,

$$\frac{\partial}{\partial u_\alpha}p_u(x,\xi)\big|_{u=0} = \langle\delta_{u_\alpha}g_u(x)\xi,\xi\rangle+\mathcal{O}(|x|^2).$$

Also, from the Taylor expansion of the generating function (12) around $s=0$, together with (13), we know

$$\phi(y,u,\eta;x) = \langle x-y,\eta\rangle - tp_u(x,\eta)+\mathcal{O}(t^2). \tag{22}$$

Besides, according to (20), $|\xi^{(j)}|^2_{g_0(x)}+V(x)=E+\mathcal{O}(\varepsilon)$, $|\eta^{(j)}|^2_{g_0(x)}+V(x)=E+\mathcal{O}(\varepsilon)$ and $|\eta'^{(j)}|^2_{g_0(x)}+V(x)=E+\mathcal{O}(\varepsilon)$ for $i\le\ell$ and $j\le q$. Therefore, for $x\in B(x_*,\tau_\varepsilon)$,

$$\frac{\partial}{\partial u_\alpha}\Phi^{[\ell,q]}\big|_{u=0} =$$
$$= -t\left(\sum_{i=1}^{\ell}\left\langle \delta_{u_\alpha} g_u(x)\xi^{(i)},\xi^{(i)}\right\rangle + \sum_{j=1}^{q}\left(\left\langle \delta_{u_\alpha} g_u(x)\eta^{(j)},\eta^{(j)}\right\rangle - \left\langle \delta_{u_\alpha} g_u(x)\eta'^{(j)},\eta'^{(j)}\right\rangle\right)\right)$$
$$+ \mathcal{O}(|x|^2) + \mathcal{O}(t^2)$$
$$= -t\, a(x)\left(\sum_{i=1}^{\ell}|\xi^{(i)}|^2_{g_0(x)} + \sum_{j=1}^{q}|\eta^{(j)}|^2_{g_0(x)} - \sum_{j=1}^{q}|\eta'^{(j)}|^2_{g_0(x)}\right) + \mathcal{O}(|x|^2) + \mathcal{O}(t^2)$$
$$= -t\, a(x)\, \ell\, (E - V(x)) + \mathcal{O}(\varepsilon) + \mathcal{O}(|x|^2) + \mathcal{O}(t^2).$$

Since $a(x) \neq 0$ for all $x \in B(x_*, \tau_\varepsilon)$, we conclude that the claim in (21) holds. We then use the operator

$$\Big(\frac{h}{i\,\partial_{u_\alpha}\Phi^{[\ell,q]}}\Big)\frac{\partial}{\partial u_\alpha}$$

to repeatedly integrate by parts in (19) and obtain

$$B_u^{[\ell,q]}(h)(\tilde{y},\tilde{z},\tilde{z}') = \mathcal{O}(h^\infty)$$

locally uniformly in $(\tilde{y},\tilde{z},\tilde{z}') \in \oplus^{\ell+2q}M$, and $u \in B^k(\varepsilon)$, and $x \in B(x_*,\tau_\varepsilon)\cap(V^{-1}(E))^c$. From (18) it follows that

$$\int_{\mathcal{B}^k(\varepsilon)} \left(\varphi_h^{(u)}(x)\right)^\ell \; \left|\varphi_h^{(u)}(x)\right|^{2q} \, du = \mathcal{O}(h^\infty),$$

locally uniformly in $x \in B(x_*,\tau_\varepsilon) \cap (V^{-1}(E))^c$.

□

4. Variance

As explained in the Introduction (see (7)), provided the perturbation is admissible at $x \in (V^{-1}(E))^c$, the case $p = 1$ in Proposition 3 shows that our random variables are semiclassically centered with

$$\mathbb{E}\left[\Re(\varphi_h^{(\cdot)}(x))\right] = \mathcal{O}(h^\infty).$$

Therefore,

$$Var\left[\Re(\varphi_h^{(\cdot)}(x))\right] = \frac{1}{|B^k(\varepsilon)|}\int_{\mathcal{B}^k(\varepsilon)} |\varphi_h^{(u)}(x)|^2\, du + \mathcal{O}(h^\infty).$$

It then follows that studying the variance is equivalent to understanding the behavior of the right hand side in the previous equality. We will need

Proposition 5. *Let g_u be admissible at $x_* \in M$. For $\varepsilon > 0$ and $|t|$ small, there exists $\tau_\varepsilon > 0$, a choice of coordinates $u'' \in \mathcal{B}^{k-n}(\varepsilon)$ of u, and an operator $A_{x,u''}(h) \in \Psi^{0,-\infty}_{cl}(M)$ defined for all $(x, u'') \in B(x_*, \tau_\varepsilon) \times \mathcal{B}^{k-n}(\varepsilon)$ making*

$$\int_{\mathcal{B}^k(\varepsilon)} \left|\varphi_h^{(u)}(x)\right|^2 du = \int_{\mathcal{B}^{k-n}(\varepsilon)} \left\langle A_{x,u''}(h)(\varphi_\hbar), \varphi_\hbar \right\rangle_{L^2(M)} du'' + \mathcal{O}(h^\infty). \tag{23}$$

In addition, there exists a constant $C_1 = C_1(\varepsilon, t, E, g_0) > 0$ so that

$$|\sigma_0\left(A_{x,u''}(h)\right)(y,\eta)| > \frac{C_1}{2} > 0 \tag{24}$$

uniformly for $(y, \eta) \in p_0^{-1}(E)$ and $(x, u'') \in B(x_*, \tau_\varepsilon) \times \mathcal{B}^{k-n}(\varepsilon)$.

Proof. Let $0 < \tau_\varepsilon < inj(M)$ be so that the admissibility condition (A) holds on $B(x_*, \tau_\varepsilon)$. That is, there exist $c > 0$ and some subset of n coordinates of u, which we denote $u' \in \mathcal{B}^n(\varepsilon)$, so that for all $x \in B(x_*, \tau_\varepsilon)$ the matrix

$$d_{u'} d_\xi (p_u(x,\xi))\big|_{u=0} \quad \text{is invertible for} \quad (x,\xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon). \tag{25}$$

We write $u'' \in \mathcal{B}^{k-n}(\varepsilon)$ for the omitted variables and assume that the coordinates of u are ordered so that $u = (u', u'')$.

Write $W_u(h)(x, y)$ for the Schwartz kernel of $W_u(h)$, and for $u'' \in \mathcal{B}^{k-n}(\varepsilon)$ and $x \in B(x_*, \tau_\varepsilon)$ define a new family of operators

$$\hat{W}_{x,u''}(h) : C^\infty(M) \to C^\infty(\mathcal{B}^n(\varepsilon)), \tag{26}$$

with Schwartz kernels

$$\hat{W}_{x,u''}(h)(u', y) := W_u(h)(x, y), \qquad \text{for } u = (u', u'').$$

Since from (9),

$$\varphi_h^{(u)}(x) = [W_u(h)\varphi_h](x) + \mathcal{O}(h^\infty) = [\hat{W}_{x,u''}(h)\varphi_h](u') + \mathcal{O}(h^\infty),$$

we then have

$$\begin{aligned}\int_{\mathcal{B}^k(\varepsilon)} \left|\varphi_h^{(u)}(x)\right|^2 du &= \int_{\mathcal{B}^k(\varepsilon)} \left|\hat{W}_{x,u''}(h)\varphi_\hbar(u')\right|^2 du + \mathcal{O}(h^\infty) \\ &= \int_{\mathcal{B}^{k-n}(\varepsilon)} \left\langle \hat{W}_{x,u''}(h)(\varphi_\hbar), \hat{W}_{x,u''}(h)(\varphi_\hbar) \right\rangle_{L^2(\mathcal{B}^n(\varepsilon))} du'' + \mathcal{O}(h^\infty) \\ &= \int_{\mathcal{B}^{k-n}(\varepsilon)} \left\langle A_{x,u''}(h)(\varphi_\hbar), \varphi_\hbar \right\rangle_{L^2(M)} du'' + \mathcal{O}(h^\infty).\end{aligned} \tag{27}$$

From (14), the Schwartz kernel of $\hat{W}_{x,u''}(h)$ is given by

$$\hat{W}_{x,u''}(h)(u', y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(y,u',u'',\xi;x)} a(u, y, \xi; x, h)\, d\xi + K_x(y, u), \tag{28}$$

where $|\partial_x^\alpha \partial_y^\beta K_x(y, u)| = \mathcal{O}_{\alpha,\beta}(h^\infty)$ uniformly in $(x, y, u) \in U \times V \times \mathcal{B}^k(\varepsilon)$ for $\varepsilon > 0$ small, where $U, V \subset \mathbb{R}^n$ are local coordinate charts. The amplitude $a(u, y, \xi; x, h) \sim$

$\sum_{j=0}^{\infty} a_j(u, y, \xi; x)h^j$ with $a_j(u, \cdot, \cdot; \cdot) \in C^\infty(B^k(\varepsilon), C_0^\infty(V \times \mathbb{R}^n \times U))$.

By the same argument presented in [8, Prop. 4.1], it can be shown that for $\varepsilon > 0$ and $|t|$ small enough, $\hat{W}_{x,u''}(h) \in I_{cl}^{0,-\infty}(M \times \mathcal{B}^n(\varepsilon); \Gamma_{x,u''})$ with

$$\Gamma_{x,u''} := \left\{ (u', d_{u'}S(t, u, \eta; x)), d_\eta S(t, u, \eta; x), \eta) : \ (d_\eta S(t, u, \eta; x), \eta) \in supp\, \chi_E^{(0)} \right\}, \\ \subset T^*\mathcal{B}^n(\varepsilon) \times T^*M. \tag{29}$$

where $u := (u', u'') \in \mathcal{B}^k(\varepsilon)$, for ε small.

From (22) and (25) we know there exists $C_0 > 0$ so that for all $x \in B(x_*, \tau_\varepsilon)$

$$|d_{u'} d_\eta \phi(y, u', u'', \eta; x)| = |s|^n \left(|d_{u'} d_\eta p_u(x, \eta)| + \mathcal{O}(s^2)\right) \geq C_0 |s|^n, \tag{30}$$

locally uniformly in (y, u, η) with $(x, \eta) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$. Let $\chi_E^{(u)}$ in (8) be chosen so that if $(d_\eta S(s, u, \eta; x), \eta), \eta) \in supp\, \chi_E^{(0)}$ for $u \in \mathcal{B}^k(\varepsilon)$ then $(x, \eta) \in p_0^{-1}((E - c\varepsilon, E + c\varepsilon))$ with $c > 0$ as in (30). This can be done because locally, according to (12), $d_\eta S(s, u, \eta; x) = x - s\partial_\eta p_u(x, \eta) + \mathcal{O}(t^2)$. By (29), such choice of cutoff function ensures that the non-degeneracy condition (30) holds on $\Gamma_{x,u''}$. Now, for u'' fixed, consider the map

$$(u', y, \eta) \mapsto d_\eta \phi(y, u', u'', \eta; x), \qquad (u', \tau; y, \eta) \in \Gamma_{x,u''}.$$

We claim that due to the the non-degeneracy condition (30), the Lagrangian (29) is a canonical graph. Indeed, (30) allows us to apply the Implicit Function Theorem and locally write $u' = u'(y, \eta)$ satisfying

$$u' = u'(y, \eta) \quad \text{when} \quad d_\eta \phi(y, u', u'', \eta; x) = 0.$$

Then, taking into account that

$$d_\eta \phi(y, u', u'', \eta; x) = 0 \quad \text{when} \quad y = d_\eta S(t, u', u'', \eta; x),$$

we write $(y, \eta) \in V \times \mathbb{R}^n$ as local parametrizing variables for $\Gamma_{x,u''}$ as in (29) and get:

$$\Gamma_{x,u''} = \left\{ \Big(u'(y, \eta), d_{u'}S\left(t, u'(y, \eta), u'', \eta; x\right) \ ; \ y, \eta\Big) : \ (y, \eta) \in supp\, \chi_E^{(0)} \right\}. \tag{31}$$

For $u'' \in \mathcal{B}^{k-n}(\varepsilon)$ and $x \in B(x_*, \tau_\varepsilon)$ define the operators

$$A_{x,u''}(h) : C^\infty(M) \to C^\infty(M),$$

$$A_{x,u''}(h) := \Big(\hat{W}_{x,u''}(h)\Big)^* \circ (\hat{W}_{x,u''}(h)). \tag{32}$$

Since $\hat{W}_{x,u''}(h) \in I_{cl}^{0,-\infty}(M \times \mathcal{B}^n(\varepsilon); \Gamma_{x,u''})$ and the immersed Lagrangian $\Gamma_{x,u''}$ is a canonical graph, the operator

$$A_{x,u''}(h) \in \Psi_{cl}^{0,-\infty}(M),$$

for $x \in B(x_*, \tau_\varepsilon)$ and $u'' \in \mathcal{B}^{k-n}(\varepsilon)$. Following the same argument presented in Corollary 4.2 of [8] its principal symbol can be locally written as

$$\begin{aligned}\sigma_0\left(A_{x,u''}(h)\right)(y,\eta) &= |\chi_E^{(u',u'')}(x,\eta)|^2 \frac{|d_x \pi G^t_{(u',u'')}(x,\eta)|}{|d_{u'} d_\eta S(t,x,\eta;u',u'')|} \\ &= |\chi_E^{(u',u'')}(x,\eta)|^2 \frac{|d_x \pi G^t_{(u',u'')}(x,\eta)|}{|t|^n \, |d_{u'} d_\eta \, p_{(u',u'')}(x,\eta)|}\end{aligned} \tag{33}$$

for $u' = u'(y,\eta)$ parametrizing the Lagrangian $\Gamma_{x,u''}$ regarded as a canonical graph. In particular, (24) holds.

□

4.1. **Proof of Theorem 1.** Since M is compact we choose a finite covering

$$M \subset \bigcup_{j=1}^N B_j(x_*, \tau_\varepsilon)$$

where for each $j = 1, \dots, N$ the ball $B_j(x_*, \tau_\varepsilon)$ is given by Proposition 3.

Fix $j \in \{1, \dots, N\}$ and let $x \in B_j(x_*, \tau_\varepsilon)$. To prove the first part of Theorem 1 we use that from Proposition 5, $A_{x,u''}(h) \in \Psi^{0,-\infty}_{cl}(M)$, so by L^2 boundedness there exists a constant $C_2^j = C_2^j(\varepsilon, t, E, g_0) > 0$ making

$$\langle A_{x,u''}(h)\varphi_\hbar, \varphi_\hbar \rangle_{L^2(M)} \le C_2^j$$

uniformly in $(x, u'', h) \in B_j(x_*, \tau_\varepsilon) \times \mathcal{B}^{k-n}(\varepsilon) \times (0, h_0]$. We obtain a lower bound from (24) and the weak Garding inequality,

$$\langle A_{x,u''}(h)(\varphi_\hbar), \varphi_\hbar \rangle_{L^2(M)} \ge C_1^j > 0$$

uniformly in $(x, u'', h) \in B_j(x_*, \tau_\varepsilon) \times \mathcal{B}^{k-n}(\varepsilon) \times (0, h_0]$. Therefore, from (7) and (23) one can choose positive constants C_1, C_2 making the first part of the statement of Theorem 1 hold uniformly in $x \in K$ where $K \subset (V^{-1}(E))^c$ is any compact subset.

To prove the second part of Theorem 1 regarding the odd moments we simply apply Proposition 3 in each ball $B_j(x_*, \tau_\varepsilon)$.

□

4.2. **Proof of Theorem 2.** From Proposition 5 and equation (7),

$$\begin{aligned}\lim_{h\to 0^+} Var\left[\Re\left(\varphi_h^{(\cdot)}(x)\right)\right] &= \lim_{h\to 0^+} \frac{1}{|\mathcal{B}^k(\varepsilon)|} \int_{\mathcal{B}^k(\varepsilon)} \left|\varphi_h^{(u)}(x)\right|^2 du \\ &= \lim_{h\to 0^+} \frac{1}{|\mathcal{B}^k(\varepsilon)|} \int_{\mathcal{B}^{k-n}(\varepsilon)} \left\langle A_{x,u''}(h)(\varphi_\hbar), \varphi_\hbar \right\rangle_{L^2(M)} du''.\end{aligned} \tag{34}$$

Since $(\varphi_h)_h$ is a quantum ergodic sequence,

$$\lim_{h\to 0^+} \left\langle A_{x,u''}(h)(\varphi_\hbar), \varphi_\hbar \right\rangle_{L^2(M)} = \frac{1}{|p_0^{-1}(E)|} \int_{p_0^{-1}(E)} \sigma_0(A_{x,u''}(h))(y,\eta)\, d\omega_E(y,\eta). \quad (35)$$

The first statement of Theorem 2 then follows from combining (34), (35) and the expression for the principal symbol (33).

The second statement of Theorem 2 about odd moments is a direct application of the second part of Theorem 1.

□

5. Admissible perturbations

In this section we study the geometry behind the admissibility condition and show that perturbations satisfying such conditions always exist. It is clear that one can always have perturbations satisfying part (B) of the admissibility condition. We therefore focus in this section on proving the existence of metric perturbations satisfying condition (A). The symbol $p_u : T^*M \to T^*M$ defined in (3) has the form

$$p_u(x,\xi) = \sum_{i,j=1}^n g_u^{ij}(x)\xi_i\xi_j + V(x).$$

Therefore, in geometric terms, a perturbation g_u with $u \in \mathcal{B}^k(\varepsilon)$ satisfies part (A) of the admissibility condition provided the map

$$\Omega_\xi : \mathcal{B}^k(\epsilon) \to \mathbb{R}^n, \qquad \Omega_\xi(u) := g_u^{-1}(\xi) = \left(\sum_{l=1}^n g_u^{il}(x)\xi_l\right)_{i=1,\dots,n}$$

is a submersion at $u = 0$ for all $(x,\xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$ and some $c > 0$.

Write $\mathcal{M}$ for the space of Riemannian metrics on M. For each coordinate u_s of u define the symmetric tensor $h^{u_s} := \partial_{u_s} g_u^{-1}|_{u=0}$ and write in local coordinates

$$h^{u_s} = h_{ij}^{u_s}\, dx_i \otimes dx_j, \qquad h_{ij}^{u_s} := \partial_{u_s} g_u^{ij}\big|_{u=0}. \quad (36)$$

It is straight forward to check that $\partial_{u_s}\partial_{\xi_i} p_u(x,\xi)\big|_{u=0} = 2\sum_{l=1}^n h_{li}^{u_s}(x)\,\xi_l$. Thereby, a metric perturbation satisfies condition (A) provided there exist $c > 0$ and an n-tuple $u' = (u_1, \dots, u_n)$ of coordinates of u so that for all $(x,\xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$, the matrix

$$\left(\sum_{l=1}^n h_{u_j}^{li}(x)\,\xi_l\right)_{i,j=1,\dots,n}$$

is invertible. By definition, the notion of admissibility depends on the direction, inside the space of symmetric tensors, in which g_0 is deformed. In what follows we show that the admissibility condition is directly related to performing the deformation g_u in sufficiently many volume preserving directions.

Let $\mathcal{P}$ denote the multiplicative group of positive smooth functions on M, which we refer to as *pointwise conformal deformations.* $\mathcal{P}$ acts on $\mathcal{M}$ by multiplication

$$\mathcal{P} \times \mathcal{M} \to \mathcal{M}, \qquad (p, g) \to pg.$$

Given $g_0 \in \mathcal{M}$, the orbit of g_0 under $\mathcal{P}$ denoted by $\mathcal{P} \cdot g_0$, is a closed submanifold of $\mathcal{M}$ with tangent space at g_0 given by

$$T_{g_0}(\mathcal{P} \cdot g_0) = \{v \in S^2(M) : \ \ v = f\, g_0,\ f \in C^\infty(M)\}. \tag{37}$$

Let μ be a volume form on M and define $\mathcal{N}_\mu := \{g \in \mathcal{M} : \mu = \mu_g\}$ where μ_g denotes the Riemannian volume measure associated to g. Then $\mathcal{N}_{\mu_{g_0}}$ is a submanifold of $\mathcal{M}$ with tangent space at $g \in \mathcal{M}$ given by (cf. [6])

$$T_{g_0}(\mathcal{N}_{\mu_{g_0}}) = \{v \in S^2(M) : \ \ tr_{g_0} v = 0\}. \tag{38}$$

For every metric $g \in \mathcal{M}$ the space of symmetric tensors has the pointwise orthogonal splitting

$$T_{g_0}\mathcal{M} = T_{g_0}(\mathcal{N}_{\mu_{g_0}}) \oplus T_{g_0}(\mathcal{P} \cdot g_0)$$

where every $v \in S^2(M)$ is decomposed as $v = (v - \frac{tr_{g_0} v}{n} g_0) + \frac{1}{n}(tr_{g_0} v)\, g_0$.

To prove (38), consider geodesic normal coordinates centered at $x_* \in M$. Locally, for a point x lying in a small geodesic neighborhood of x_*, one can write $g_0(x) = \delta_{ij} + \mathcal{O}(|x|^2)$ and therefore $tr_{g_0} v(x) = tr(v(x)) + \mathcal{O}(|x|^2)$. Also, if g_{ij} denotes the metric in local coordinates, then the volume form at y is determined by $\sqrt{\det g_{ij}(x)}$, hence fixing the volume form at x is equivalent to preserving the determinant, and it is well-known that condition is equivalent to perturbing by a traceless matrix $v(x)$. We remark that if a deformation preserves the volume form for a metric g on TM, then it also preserves the volume form for the corresponding metric g^{-1} on T^*M, since the latter is given by $\sqrt{\det g^{ij}} = 1/\sqrt{\det g_{ij}}$. Let g_u be a metric deformation of g_0 and continue to write $h^{u_s} = \delta_{u_s} g_u^{-1}$. Also, working in geodesic normal coordinates at x_*, it is not difficult to show that volume-preserving deformations are characterized by the condition $tr_{g_0^{-1}}(h^{u_s}(x)) = 0$. We shall show below that the admissibility condition holds for such deformations.

5.1. **Surfaces.** On surfaces, we claim that perturbations g_u that have two linearly independent u-derivatives in the volume preserving directions are admissible.

Proposition 6. *Let (M, g_0) be a compact Riemannian surface. Let E be a regular value of p_0. Suppose g_u with $u \in \mathcal{B}^k(\varepsilon)$ is a perturbation of g_0 such that there exist two coordinates $u' = (u_1, u_2)$ of u for which $h^{u_1}(x)$ and $h^{u_2}(x)$ are linearly independent tensors with $tr_{g_0^{-1}}(h^{u_1}) = tr_{g_0^{-1}}(h^{u_2}) = 0$ for all $x \in M$.*
Then, for ε small enough, the perturbation g_u satisfies part (A) *of the admissibility condition at every $x \in (V^{-1}(E))^c$.*

Proof. By assumption, $tr_{g_0^{-1}}(h^{u_s}) = 0$ for $s = 1, 2$. Let $x_* \in M$ be such that x belongs to a geodesic ball cantered at x_*, and consider normal coordinates at x_*. In these coordinates, $g_{0_{ij}}(x) = \delta_{ij} + \mathcal{O}(|x|^2)$ for x being at a small distance $|x|$ from x_*. Therefore, $h^{u_s}_{11}(x) = -h^{u_s}_{22}(x) + \mathcal{O}(|x|^2)$ for $s = 1, 2$. It is straight forward to check

$$\det\left(\sum_{j=1}^{n} h_{ij}^{u_s}(x)\,\xi_j\right)_{s,i=1,2} = |\xi|^2_{g_0(x)}\left(\det\begin{pmatrix} h_{11}^{u_1}(x) & h_{11}^{u_2}(x)\\ h_{12}^{u_1}(x) & h_{12}^{u_2}(x)\end{pmatrix} + \mathcal{O}(|x|^2)\right).$$

Since we are only interested in what happens when $|\xi|^2_{g_0(x)} + V(x) = E + \mathcal{O}(\varepsilon)$, the result follows from the assumption $V(x) \neq E$ and the fact that $h^{u_1}(x)$ and $h^{u_2}(x)$ are linearly independent tensors. □

5.2. **Manifolds.** In what follows we show that on an n-dimensional manifold we can always have admissible perturbations.

Let M be an n-dimensional compact manifold and fix $x_* \in M$. Consider a geodesic normal coordinate system at x_*; for $x \in B(x_*, inj(M))$ we have $g_{0ij}(x) = \delta_{ij} + \mathcal{O}(|x|^2)$. We shall consider deformations of the reference metric g_0 that, as in the surface case, preserve the volume form. Infinitesimally, as explained in (38), the corresponding quadratic form is given by a traceless symmetric matrix. The space of traceless symmetric tensors at x has dimension

$$\kappa_n := \frac{n^2+n-2}{2}, \tag{39}$$

and the basis of the space of such forms is given by

$$\xi_1^2 - \xi_i^2,\ \ 2 \leq i \leq n; \quad \text{and} \quad \xi_j\xi_k,\ \ 1 \leq j < k \leq n,$$

for $\xi = (\xi_1, \dots, \xi_n) \in T_x^*M$. Denote the corresponding symmetric tensors by $h_i(x)$ with $i = 1, \dots, \kappa_n$.

Proposition 7. *The general perturbation of g_u with $u \in \mathcal{B}^n(\varepsilon)$ defined by*

$$g_u^{-1}(x) = g_0^{-1}(x) + \sum_{i=1}^{\kappa_n} u_i\, h_i(x)$$

satisfies part (A) *of the admissibility condition at every $x \in \mathcal{B}(x_*, inj(M))$.*

Proof. We assume that the basis $h_1, \dots, h_{\kappa_n}$ is L^2-normalized on the sphere $S^{n-1} = \{\xi : |\xi|_{g(x)} = 1\}$. Clearly, $\partial_{u_j}(|\xi|_{g_u(x)}) = h_j(\xi)$, and hence the j-th column (say) of the "mixed hessian" matrix corresponds to the gradient $dh_j(\xi)$.

Now, since the sphere S^{n-1} is a homogeneous space, the round metric $g_{S^{n-1}}$ is a critical metric for the corresponding eigenvalue functional $\lambda(g) \cdot \mathrm{Vol}(g)^{2/n}$, where λ denotes the second positive eigenvalue (without multiplicity) of the Laplacian.

It is well-known ([7, 12, 14]) that for such metrics, the L^2-normalized basis of the eigenspace $E(\lambda)$ (which can be chosen as $\{h_1, h_2, \dots, h_{\kappa_n}\}$ in our case) satisfies

$$\sum_{j=1}^{\kappa_n} dh_j \otimes dh_j = c\lambda\, g_{S^{n-1}}, \qquad c \neq 0.$$

Assume for contradiction that the subspace spanned by $dh_1(x), dh_2(x), \dots, dh_{\kappa_n}(x)$ has less than the full dimension $n-1$ in $T_x^*(S^{n-1})$ at some point $x \in S^{n-1}$. Then it is easy to see that the quadratic form $\sum_{j=1}^{\kappa_n} dh_j \otimes dh_j$ will have rank strictly smaller than the full rank $n-1$ at x (the corresponding matrix will have an eigenvalue 0).

However, the round metric $g_{S^{n-1}}$ on the sphere clearly has the full rank at every point on S^{n-1}. The contradiction shows that $\{dh_j(x), 1 \leq j \leq \kappa_n\}$ span the full $T_x^*(S^{n-1})$, which proves the required non-degeneracy condition. $\square$

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