Extremal metrics for λ_1

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J-L-Na-Ni-P: "How large can the first eigenvalue be on a surface of genus two?" IMRN 2005, 3967-3985.

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Preliminaries

(M_γ, g) closed surface of genus γ with metric g, Δ_g - Laplacian. Spectrum: $\Delta\phi_i = \lambda_i\phi_i$,

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

Question: How large can λ_1 be on M_γ? We consider upper bounds on λ_1 depending on the *topology* and the *area* of the surface.

What is known:

$$\lambda_1 \cdot Area(M_\gamma) \leq 8\pi \left[\frac{\gamma+3}{2}\right]$$

for orientable M (Hersch '70, Yang-Yau '80),

$$\lambda_1 \cdot Area(M_\gamma) \leq 24\pi \left[\frac{\gamma+3}{2}\right]$$

for non-orientable M (Li-Yau '82). In genus 0:

$$\lambda_1 \cdot Area(\mathbf{S}^2) \leq 8\pi, \quad \lambda_1 \cdot Area(\mathbf{RP}^2) \leq 12\pi.$$

Equalities achieved on round metrics.

Problem: sharp upper bounds on λ_1 for genus $\gamma \geq 1$. Apriori, methods of Hersch-Yang-Yau-Li do not provide *sharpness.* How to find $\sup_g \lambda_1 \cdot Area$? Is it attained on a smooth metric?

Remark: If dimension is $n \geq 3$,

$$\sup \lambda_1 \cdot Volume^{2/n} = \infty$$

Definition. A metric g on a surface is λ_1-*maximal* if for any metric $\tilde{g}$ of the same area $\lambda_1(g) \geq \lambda_1(\tilde{g})$. A λ_1-maximal metric is global maximum of the functional

$$\lambda_1 : g \to \mathbf{R}_+$$

Consider critical points of this functional. They are called *extremal metrics.* g_t - analytic deformation of g_0. Metric g_0 - extremal iff

$$\frac{d}{dt}\lambda_1|_{t=0^+}, \quad \frac{d}{dt}\lambda_1|_{t=0^-}$$

have opposite signs.

Properties of extremal metrics:

- $mult(\lambda_1) \geq 3$, equality only on $(\mathbf{S}^2, st)$.
- a surface with an extremal metric admits a *minimal isometric immersion* by the first eigenfunctions into a sphere of certain dimension (Nadirashvili, '96).

To find extremal metrics - study minimal immersions into spheres.

Remark: Similar results for metrics on graphs were obtained in [J-R].

Examples of extremal metrics:
1) $\mathbf{S}^2$, round metric ($\longrightarrow S^2$)
2) $\mathbf{RP}^2$, round metric ($\longrightarrow S^4$)
3) $\mathbf{T}^2$, flat equilateral torus ($\longrightarrow S^5$)
4) $\mathbf{T}^2$, flat square torus ($\longrightarrow S^3$)

1–3 are λ_1-maximal, 4 is a saddle. Maximality of 3) is Berger's conjecture, plan of the proof proposed by Nadirashivili '96 (cf. talk of Girouard!) There are no other extremal metrics on $\mathbf{S}^2$, $\mathbf{RP}^2$, $\mathbf{T}^2$. (El Soufi-Ilias, '00).

What happens on other surfaces? We study the **Klein bottle** and the **surface of genus** 2.

Theorem (J-Na-P) An $\mathbf{S}^1$-equivariant metric g_0 given by

$$\frac{9+(1+8\cos^2 v)^2}{1+8\cos^2 v}\left(du^2+\frac{dv^2}{1+8\cos^2 v}\right),$$

$0 \leq u < \pi/2$, $0 \leq v < \pi$, is an extremal metric on a Klein bottle $\mathbf{K}$. The surface $(\mathbf{K}, g_0)$ admits a minimal isometric embedding into $\mathbf{S}^4$ by the first eigenfunctions. λ_1 has multiplicity 5 and

$$\lambda_1 \cdot Area(K, g_0) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right),$$

$$\text{where} \quad E(T) = \int_0^{\pi/2} \sqrt{1-T^2\sin^2\alpha}\, d\alpha$$

is a complete elliptic integral of 2nd kind.

Theorem (J-Na-P/El Soufi-Giacomini-Jazar) The metric g_0 is the unique extremal metric on the Klein bottle.

Remark: The metric $(\mathbf{K}, g_0)$ has *variable* curvature, unlike other examples of extremal metrics. It is a *bipolar* (dual) surface for a Lawson torus (a minimally immersed torus in $\mathbf{S}^3$). Metric g_0 realizes the maximal possible multiplicity of λ_1 on a Klein bottle. All known λ_1-maximal metrics maximize multiplicity of λ_1.

Genus 2**:** Yang-Yau $\Rightarrow$

$$\lambda_1 \text{Area}(\mathcal{P}) \leq 16\pi.$$

Conjecture (J-L-Na-Ni-P): The upper bound of Yang-Yau is sharp in genus 2. This bound is attained on a singular surface which is realized as a double branched covering of the round sphere, with six doubly ramified points located at the vertices of the octahedron (at the intersection of S^2 with the coordinate axes). This surface has a conformal type of the *Bolza* surface $w^2 = z^5 - z$.

It is known that Bolza surface has the largest symmetry group of all Riemann surfaces of genus 2.

Proofs:

Klein bottle: Study the minimal immersions into S^4 by first eigenfunctions, reduce to a completely integrable system of ODE-s, prove that there exists a unique periodic solution with required initial conditions and period.

Genus 2: Study *even* and *odd* spectrum on the surface with respect to the hyperelliptic involution. $\lambda_1^{even} = 2$ (same as S^2). Need to show $\lambda_1^{odd} \geq 2$ ($\Rightarrow$ Conjecture).

λ_1^{odd} is equal to the first eigenvalue in certain mixed Dirichlet-Neumann boundary value problem on a hemisphere. Numerically, $\lambda_1^{odd} > 2.26$.