Extremal metrics for $\lambda_1$
Dmitry Jakobson (McGill)

www.math.mcgill.ca/jakobson

Joint with M. Levitin, N. Nadirashvili, N. Nigam, I. Polterovich and I. Rivin


J-L-Na-Ni-P: “How large can the first eigenvalue be on a surface of genus two?” IMRN 2005, 3967-3985.

Preliminaries

$(M_\gamma, g)$ closed surface of genus $\gamma$ with metric $g$, $\Delta_g$ - Laplacian. Spectrum: $\Delta \phi_i = \lambda_i \phi_i$,

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

**Question:** How large can $\lambda_1$ be on $M_\gamma$? We consider upper bounds on $\lambda_1$ depending on the topology and the area of the surface.

**What is known:**

$$\lambda_1 \cdot \text{Area}(M_\gamma) \leq 8\pi \left[\frac{\gamma + 3}{2}\right]$$

for orientable $M$ (Hersch ’70, Yang-Yau ’80),

$$\lambda_1 \cdot \text{Area}(M_\gamma) \leq 24\pi \left[\frac{\gamma + 3}{2}\right]$$

for non-orientable $M$ (Li-Yau ’82). In genus 0:

$$\lambda_1 \cdot \text{Area}(S^2) \leq 8\pi, \quad \lambda_1 \cdot \text{Area}(\mathbb{RP}^2) \leq 12\pi.$$ 

Equalities achieved on round metrics.
**Problem:** sharp upper bounds on $\lambda_1$ for genus $\gamma \geq 1$. Apriori, methods of Hersch-Yang-Yau-Li do not provide sharpness. How to find $\sup_g \lambda_1 \cdot \text{Area}$? Is it attained on a smooth metric?

**Remark:** If dimension is $n \geq 3$,

$$\sup \lambda_1 \cdot \text{Volume}^{2/n} = \infty$$

**Definition.** A metric $g$ on a surface is $\lambda_1$-maximal if for any metric $\tilde{g}$ of the same area $\lambda_1(g) \geq \lambda_1(\tilde{g})$. A $\lambda_1$-maximal metric is global maximum of the functional

$$\lambda_1 : g \rightarrow \mathbb{R}_+$$

Consider critical points of this functional. They are called extremal metrics. $g_t$ - analytic deformation of $g_0$. Metric $g_0$ - extremal iff

$$\frac{d}{dt} \lambda_1 \big|_{t=0^+}, \quad \frac{d}{dt} \lambda_1 \big|_{t=0^-}$$

have opposite signs.
Properties of extremal metrics:

- \( \text{mult}(\lambda_1) \geq 3 \), equality only on \((S^2, st)\).
- a surface with an extremal metric admits a *minimal isometric immersion* by the first eigenfunctions into a sphere of certain dimension (Nadirashvili, ’96). To find extremal metrics - study minimal immersions into spheres.

**Remark:** Similar results for metrics on graphs were obtained in [J-R].

**Examples of extremal metrics:**
1) \( S^2 \), round metric (\( \rightarrow S^2 \))
2) \( \mathbb{RP}^2 \), round metric (\( \rightarrow S^4 \))
3) \( T^2 \), flat equilateral torus (\( \rightarrow S^5 \))
4) \( T^2 \), flat square torus (\( \rightarrow S^3 \))

1–3 are \( \lambda_1 \)-maximal, 4 is a saddle. Maximality of 3) is Berger’s conjecture, plan of the proof proposed by Nadirashvili ’96 (cf. talk of Girouard!) There are no other extremal metrics on \( S^2 \), \( \mathbb{RP}^2 \), \( T^2 \). (El Soufi-Ilias, ’00).
What happens on other surfaces? We study the **Klein bottle** and the **surface of genus** 2.

**Theorem (J-Na-P)** An $S^1$-equivariant metric $g_0$ given by

$$\frac{9 + (1 + 8 \cos^2 v)^2}{1 + 8 \cos^2 v} \left( du^2 + \frac{dv^2}{1 + 8 \cos^2 v} \right),$$

$0 \leq u < \pi/2$, $0 \leq v < \pi$, is an extremal metric on a Klein bottle $K$. The surface $(K, g_0)$ admits a minimal isometric embedding into $S^4$ by the first eigenfunctions. $\lambda_1$ has multiplicity 5 and

$$\lambda_1 \cdot \text{Area}(K, g_0) = 12\pi E \left( \frac{2\sqrt{2}}{3} \right),$$

where

$$E(T) = \int_0^{\pi/2} \sqrt{1 - T^2 \sin^2 \alpha} \, d\alpha$$

is a complete elliptic integral of 2nd kind.

**Theorem (J-Na-P/El Soufi-Giacomini-Jazar)** The metric $g_0$ is the unique extremal metric on the Klein bottle.
Remark: The metric \((K, g_0)\) has \textit{variable} curvature, unlike other examples of extremal metrics. It is a \textit{bipolar} (dual) surface for a Lawson torus (a minimally immersed torus in \(S^3\)). Metric \(g_0\) realizes the maximal possible multiplicity of \(\lambda_1\) on a Klein bottle. All known \(\lambda_1\)-maximal metrics maximize multiplicity of \(\lambda_1\).
**Genus 2:** Yang-Yau ⇒

\[ \lambda_1 \text{Area}(\mathcal{P}) \leq 16\pi. \]

**Conjecture (J-L-Na-Ni-P):** The upper bound of Yang-Yau is sharp in genus 2. This bound is attained on a singular surface which is realized as a double branched covering of the round sphere, with six doubly ramified points located at the vertices of the octahedron (at the intersection of \( S^2 \) with the coordinate axes). This surface has a conformal type of the Bolza surface \( w^2 = z^5 - z \).

It is known that Bolza surface has the largest symmetry group of all Riemann surfaces of genus 2.
Proofs:

**Klein bottle**: Study the minimal immersions into $S^4$ by first eigenfunctions, reduce to a completely integrable system of ODEs, prove that there exists a unique periodic solution with required initial conditions and period.

**Genus 2**: Study even and odd spectrum on the surface with respect to the hyperelliptic involution. $\lambda_1^{even} = 2$ (same as $S^2$). Need to show $\lambda_1^{odd} \geq 2 \ (\Rightarrow \ Conjecture)$.

$\lambda_1^{odd}$ is equal to the first eigenvalue in certain mixed Dirichlet-Neumann boundary value problem on a hemisphere. Numerically, $\lambda_1^{odd} > 2.26$. 