Scalar and Q-curvature of random Riemannian metrics

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- (M, g) is n-dimensional compact manifold, $n \geq 2$.
- Goal: study scalar curvature R of *random* Riemannian metrics on M. We start with Gauss curvature K in dimension $n = 2$; $R = 2K$.
- *Scalar curvature:* Geometric meaning: as $r \to 0$,

$$\mathrm{vol}(B_M(x_0, r)) = \mathrm{vol}(B_{\mathbf{R}^n}(r)) \left[1 - \frac{R(x_0) r^2}{6(n+2)} + O(r^4)\right].$$

- Uniformization theorem in dimension 2: in every conformal class, there exists a unique metric of constant scalar curvature R_0. $R_0 > 0$ for $M = S^2$, $R_0 = 0$ for $M = \mathbf{T}^2$, and $R_0 < 0$ for surfaces of genus $\gamma \geq 2$.

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- **Questions:** What is the *probability* that a random metric g_1 in a conformal class has *non-vanishing* curvature R_1, $M \neq \mathbf{T}^2$? or that it satisfies certain curvature bounds?
- Use *Laplacian* to define random metrics in a *conformal class* and to estimate that probability.
- Techniques: differential geometry; spectral theory of elliptic operators; Gaussian random fields on manifolds (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).

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- g_0 - reference metric on M. Conformal class of g_0: $\{g_1 = e^f \cdot g_0\}$; f is a random (suitably regular) function on M.
- Δ_0 - Laplacian of g_0. Spectrum: $\Delta_0 \phi_j + \lambda_j \phi_j = 0, \;\; 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Define f by

$$f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x),$$

$a_j \sim \mathcal{N}(0,1)$ are i.i.d standard Gaussians, $c_j = F(\lambda_j) \to 0$ (*damping*):
- $c_j = \lambda_j^{-s}$ (random *Sobolev* metric); $c_j = e^{-t\lambda_j}$ (random *real analytic* metric).
- The *covariance function* $r_f(x,y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y)$, for $x, y \in M$.
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- Sobolev regularity:
  Proposition 1: If $c_j = O(\lambda_j^{-s}), s > n/2$, then $f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, then $f \in C^2$ a.s.
- **Volume change:** Let $V_0 = \mathrm{vol}(M, g_0)$. If $g_1 := g_1(a) = e^{af} g_0$, then $dV_1 = e^{naf/2} dV_0$. One can show that $\lim_{a \to 0} \mathbb{E}[\mathrm{vol}(M, g_1(a))] = V_0$.

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- Let $n = 2$, and $g_1 = e^{af} g_0$. Then

$$R_1 = e^{-af}[R_0 - ah] \quad (1)$$

$M \neq \mathbf{T}^2$. Estimate the probability of

$$\{\mathrm{Sgn}(R_1) = \mathrm{Sgn}(R_0)\}$$

- **Observation:** If $R_0 \neq 0$, then $\mathrm{Sgn}(R_1) = \mathrm{Sgn}(R_0)\mathrm{Sgn}(1 - a\Delta_0 f / R_0)$.
- Let $P(a) := \mathrm{Prob}\{\exists x : \mathrm{Sgn} R_1(x) = \mathrm{Sgn} R_0\}$, or $P(a) = \mathrm{Prob}\{\exists x \in M : 1 - a(\Delta_0 f)(x)/R_0(x) < 0\}$. Then

$$P(a) = \mathrm{Prob}\{\sup_{x \in M}(\Delta_0 f)(x)/R_0(x) > 1/a\},$$

Consider the random field $v = (\Delta_0 f)/R_0$. Then

$$r_v(x, y) = \frac{\sum_j (c_j \lambda_j)^2 \phi_j(x)\phi_j(y)}{R_0(x)R_0(y)}.$$

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- We shall estimate $P(a)$ in the limit $a \to 0$. Geometrically, this implies that a.s. $g_1(a) \to g_0$, so $P(a) \to 0$. We want to estimate the *rate*.
- First use **Proposition 2** (Borell, TIS, 1975-76): Let v be a centered Gaussian process, a.s. bounded on M, and $\sigma_v^2 := \sup_{x \in M} \mathbb{E}[v(x)^2]$. Let $||v|| := \sup_{x \in M} v(x)$; then $E\{||v||\} < \infty$, and $\exists \alpha$ so that for $\tau > E\{||v||\}$ we have

$$\mathrm{Prob}\{||v|| > \tau\} \leq e^{\alpha\tau - \tau^2/(2\sigma_v^2)}.$$

- Assume that $R_0 \in C^0$, $s > 2$, then $v \in C^0(M)$ a.s. and Proposition 2 applies. In our situation, $\tau = 1/a \to \infty$ as $a \to 0$, so $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma_v^2)]$.

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- To estimate $P(a)$ *from below* choose $x_0 \in M$ where the variance $r_v(x,x)$ attains its supremum σ_v^2. Clearly, $\mathrm{Prob}(||v|| > 1/a) \geq \mathrm{Prob}(v(x_0) > 1/a) = \frac{1}{\sqrt{2\pi}} \int_{1/(a\sigma_v)}^{\infty} e^{-t^2/2} dt$. Combine the estimates:
- **Theorem 3:** Assume that $R_0 \in C^0, c_j = O(\lambda_j^{-s}), s > 2$. Then $\exists C_1 > 0, C_2 > 0$ such that

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as $a \to 0$. In particular $\lim_{a\to 0} a^2 \ln P(a) = \frac{-1}{2\sigma_v^2}$.

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- **Random real-analytic metrics.** Choose the coefficients $c_j = e^{-\lambda_j T/2}/\lambda_j$. Then

$$r_v(x, x, T) = e^*(x, x, T)/(R_0(x))^2.$$

where $e^*(x, x, T)$ is the heat kernel, *without the constant term.*

- **Small T asymptotics** of $e^*(x, x, T)$ imply that as $T \to 0^+$,

$$\sigma_v^2 \sim \frac{1}{(4\pi T)^{n/2} \inf_{x \in M}(R_0(x))^2}.$$

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- **Theorem 4.** $n = 2, M \neq \mathbf{T}^2$. Let g_0 and g_1 have equal areas, R_0 and R_1 have constant sign, $R_0 \equiv const$ and $R_1 \not\equiv const$. Then $\exists a_0 > 0, T_0 > 0$ (that depend on g_0, g_1) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $P(a, T, g_1) > P(a, T, g_0)$.
- **Proof:** By Gauss-Bonnet, $\int_M R_0 dV_0 = \int_M R_1 dV_1$. Since $\mathrm{A}(M, g_0) = \mathrm{A}(M, g_1)$; and since $R_0 \equiv const$ and $R_1 \not\equiv const$, it follows that $b_0 := \min_{x \in M}(R_0(x))^2 > \min_{x \in M}(R_1(x))^2 := b_1$. Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma_v^2(g_1, T)}{\sigma_v^2(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$

The result follows easily from Theorem 3.

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- **Large T asymptotics:**
  λ_1 - the smallest nonzero eigenvalue of $-\Delta_0$. Let $m = m(\lambda_1)$ be the multiplicity of λ_1, and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^{m} \phi_j(x)^2}{R_0(x)^2}. \qquad (2)$$

- One can show that

$$\lim_{T\to\infty} \frac{\sigma_v^2(T)}{F e^{-\lambda_1 T}} = 1.$$

- **Theorem 5.** Let g_0 and g_1 be two metrics (of equal area) on a compact surface M, such that R_0 and R_1 have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on g_0, g_1), such that for all $a < a_0$ and $T > T_0$ we have $P(a, T; g_0) < P(a, T; g_1)$.

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- To summarize: Small $T \Rightarrow$ metrics with $R_0 \equiv$ *const* extremal.
- Large $T \Rightarrow$ metrics with the largest λ_1 extremal.
- Genus 0: (S^2, *round*) extremal for *both* small T and large T (Hersch). **Conjecture:** extremal for *all* T.
- Genus $\gamma \geq 2$: Small $T \Rightarrow$ hyperbolic metrics extremal.
- Large T: By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize λ_1 in their conformal class.
- Genus 2: Metrics maximizing λ_1 for surfaces of genus 2 of fixed area are branched coverings of the round S^2 (J, Levitin, Nigam, Nadirashvili, Polterovich).
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- $\mathcal{E}_m$ - space of spherical harmonics of degree m, dimension $N_m = 2m+1$; the corresponding eigenvalue is $E_m = m(m+1)$. Let $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ be an orthonormal basis of $\mathcal{E}_m$.
- Let $f(x) = -\sqrt{|\mathcal{S}^2|} \sum_{m\geq 1,\, k} \frac{\sqrt{c_m}}{E_m\sqrt{N_m}} a_{m,k}\eta_{m,k}(x)$, where $a_{m,k}$ are standard Gaussian i.i.d. and $c_m > 0$ are (suitably decaying) constants satisfying $\sum_{m=1}^{\infty} c_m = 1$.
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- We next estimate the probability of the event $\{||R_1 - R_0||_\infty < u\}, u > 0$; we shall do that for $g_1 = e^{af} g_0$, in the limit $a \to 0$. The result below hold for any compact orientable surface, including $\mathbf{T}^2$.
- To state the result, we define a new random field w on M:

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has a *gradient term* $-a^2(n-1)(n-2)|\nabla_0 f|^2/4$, that *vanished* for $n = 2$. Accordingly, the random field $R_1 e^{af}$ is no longer Gaussian, making its study more difficult.
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Assume that $\forall x \in M. R_0(x) < 0$. Then there exists $\alpha > 0$ so that

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$$\kappa = \frac{4\sigma_v^2(n-1)}{\sigma_2 n(n-2)}.$$

- In dimension $n > 3$, after a conformal change of variables, Laplacian acquires a gradient term. Problem: construct (possibly higher order) elliptic operators so that after a conformal change of variables, the gradient term vanishes.
- Example: $n = 4$; *Paneitz operator*

$$P_4 = \Delta_g^2 + \delta[(2/3)R_g g - 2\mathrm{Ric}_g]d.$$

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- $P = \Delta^{n/2} + \textit{lower order terms}$. P is self-adjoint (Graham, Zworski, Fefferman). Under a conformal transformation of metric $\tilde{g} = e^{2\omega} g$, the operator P changes as follows: $\widetilde{P} = e^{-n\omega} P$. No lower order terms!
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- Important properties of Q-curvature: it is equal to $1/(2(n-1))\Delta^{n/2}R$ modulo nonlinear terms in curvature. Under a conformal transformation of variables $\tilde{g} = e^{2\omega}g$ on M^n, the Q-curvature transforms as follows:
$$P\omega + Q = \tilde{Q}e^{n\omega}. \quad (3)$$
Integral of the Q-curvature is conformally invariant.
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- **Proposition 9:** (M, g) compact, $n \geq 4$ even, Assume that M satisfies the following "generic" assumptions:
  i) $n = 4$: $\ker P_n = \{const\}$, and $\int_M Q dV \neq 8\pi^2 k, k = 1, 2, \dots$
  ii) $n \geq 6$: $\ker P_n = \{const\}$, and $\int_M Q dV \neq (n-1)!\omega_n k, k = 1, 2, \dots$, where $(n-1)!\omega_n = \int_{S^n} Q dV$, the integral of Q-curvature for the round $\mathcal{S}^n$.

  Then there exists a metric g_Q on M in the conformal class of g with constant Q-curvature. If $n = 4, \int_M Q dV < 8\pi^2$, $P_4 \geq 0$ and $\ker P_4 = \{const\}$, then g_Q is unique.
- If g has positive R and $M \neq \mathcal{S}^4$, then the assumption $\int_M Q dV < 8\pi^2$ is satisfied; if in addition $\int_M Q \geq 0$, then the assumptions $P_4 \geq 0$ and $\ker P_4 = \{const\}$ are also satisfied.

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- It is possible to generalize Theorems 3, 7 for Q-curvature:
- Strategy:
  i) Consider (M, g_0) such that Q_0 has constant sign;
  ii) Consider the conformal perturbation $g_1 = e^{2af} g_0$ where a is a positive number; expand f in a series of eigenfunctions of P_n.
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- Improve estimates for the scalar curvature in higher dimensions.
- Consider "rough" metrics that arise in 2D quantum gravity.
- Study the case when $a \nrightarrow 0$.
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class.
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
- Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
- Δ: small eigenvalues, heat kernel asymptotics.
- Eigenfunctions: prove for "generic" metrics results that seem difficult (or wrong!) for *all* metrics.
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