SCALAR CURVATURE AND Q-CURVATURE OF RANDOM METRICS.

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Abstract. We study Gauss curvature for random Riemannian metrics on a compact surface, lying in a fixed conformal class; our questions are motivated by comparison geometry. Next, analogous questions are considered for the scalar curvature in dimension $n > 2$, and for the Q-curvature of random Riemannian metrics.

1. Introduction

The goal of the authors in this paper is to investigate standard questions in *comparison geometry* for random Riemannian metrics lying in the same conformal class.

Random metrics have long been considered in 2-dimensional conformal field theory and quantum gravity, random surface models and other fields. In addition, random metrics are frequently considered in and cosmology and astrophysics, in the study of gravitational waves and cosmic microwave background radiation.

Random metrics lying in a fixed conformal class are easiest to treat analytically; in addition, many classical problems in differential geometry are naturally formulated and solved for metrics lying in a fixed conformal class (uniformization theorem for Gauss curvature in dimension 2, Yamabe problem and uniformization problem for Q-curvature in higher dimensions). Accordingly, it is natural to consider random metrics lying in a fixed conformal class.

The questions considered in this paper are motivated by *comparison geometry*. Since the 19th century, many results have been established comparing geometric and topological properties of manifolds where the (sectional or Ricci) curvature is bounded from above or from below, with similar properties of manifolds of constant curvature. Examples include Toponogov Theorem (comparing triangles); sphere theorems of Myers and Berger-Klingenberg for positively-curved manifolds; volume of the ball comparison theorems of Gromov and Bishop; splitting theorem of Cheeger and Gromoll; Gromov's pre-compactness theorem; theorems about geodesic flows and properties of fundamental group for negatively-curved manifolds; and numerous other results.

When studying such questions for random Riemannian metrics, the first natural question is to estimate the *probability* of the metric satisfying certain curvature

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2000 *Mathematics Subject Classification.* Primary: 60G60 Secondary: 53A30, 53C21, 58J50, 58D17, 58D20.

Key words and phrases. Comparison geometry, conformal class, scalar curvature, Q-curvature, Gaussian random fields, excursion probability, Laplacian, conformally covariant operators.

D.J. was supported by NSERC, FQRNT and Dawson fellowship.

I.W. was supported by a CRM-ISM fellowship, Montréal and the Knut and Alice Wallenberg Foundation, grant KAW.2005.0098.

bounds, in a suitable regime. The present paper addresses such questions for *scalar curvature*, and also for Branson's *Q-curvature*.

Let (M, g) be an n-dimensional compact manifold, $n \geq 2$. Recall that the *Riemann curvature tensor* is defined by $R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, where ∇ denotes the Levi-Civita connection. In local coordinates,

$$R_{ijkl} := \langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle.$$

The *Ricci curvature* of g can be defined in local coordinates by the formula $R_{jk} = g^{il} R_{ijkl}$. In geodesic normal coordinates, the volume element takes the form $dV_g = [1 - (1/6)R_{jk}x^j x^k + O(|x|^3)]dV_{Euclidean}$.

The *Scalar curvature* can be defined in local coordinates by the formula $R = g^{ik}R_{ik}$. Geometrically, $R(x_0)$ determines the difference of the volume of a small ball of radius r in M (centered at x_0) and the Euclidean ball of the same radius: as $r \to 0$,

$$\mathrm{vol}(B_M(x_0, r)) = \mathrm{vol}(B_{\mathbf{R}^n}(r)) \left[1 - \frac{R(x_0)r^2}{6(n+2)} + O(r^4) \right].$$

We study the behavior of scalar curvature for random Riemannian metrics in a fixed conformal class, where the conformal factor is a random function possessing certain smoothness. We consider random metrics that are close to a "reference" metric that we denote g_0.

The paper addresses two main questions:

Question 1.1. *Assuming that the scalar curvature R_0 of the reference metric g_0 doesn't vanish, what is the probability that the scalar curvature of the perturbed metric changes sign?*

We remark that in each conformal class, there exists a *Yamabe metric* with constant scalar curvature $R_0(x) \equiv R_0$, [Yam, Au76, Sch84, Tr]; the sign of R_0 is uniquely determined. Problem (i) can be posed in each conformal class where $R_0 \neq 0$ (e.g. in dimension two, for $M \not\cong \mathbf{T}^2$). Also, it was shown in [CY, DM, N] that in every conformal class satisfying certain *generic* conditions, there exists a metric g_0 with constant Q-curvature, $Q_0(x) \equiv Q_0$. Question 1.1 can be posed in each conformal class where $Q_0 \neq 0$.

In dimension 2, this problem is studied in section 3 for a.s. C^0 metrics on surfaces S_γ of genus $\gamma \neq 1$ (if the "reference" metric has scalar curvature of constant sign). The probability estimates are greatly improved in section 4 for a.s. C^2 metrics on $\mathcal{S}^2$. Problem (i) is addressed for scalar curvature in higher dimensions in section 6; and for Q-curvature in section 7.6. In section 3.1 several comparison theorems are proved for random real-analytic metrics.

Question 1.2. *What is the probability that curvature of the perturbed metric changes by more than u (where u is a positive real parameter, subject to some restrictions)?*

Question 1.2 is studied on surfaces in section 5, and for Q-curvature in section 7.7.

In appendix B, we include a short survey of the results on Yamabe problem, and in appendix A a short survey on existence of metrics of positive and negative scalar curvature in conformal classes.

Finally, in appendix C we verify the assumptions needed to apply the results of R. Adler and J. Taylor to answer Question (i) on the round metric on S^2.

Our techniques are inspired by [AT03, ATT05, AT08, Bl].

1.1. **Conventions.** Given a random field $F : T \to \mathbb{R}$ on parameter set T we define the random variable

$$\|F\|_T := \sup_{t \in T} F(t).$$

Note that there is no absolute value in the definition of $\|\cdot\|_T$, so that it is by no means a norm; this is in contrast to $\|\cdot\|_\infty$, which denotes the sup norm. Let $\Psi(u)$ denote the error function

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-t^2/2} dt.$$

1.2. **Acknowledgement.** The authors would like to thank R. Adler, P. Guan, V. Jaksic, N. Kamran, S. Molchanov, I. Polterovich, G. Samorodnitsky, B. Shiffman, J. Taylor, J. Toth, K. Worsley and S. Zelditch for stimulating discussions about this problem. The authors would like to thank for their hospitality the organizers of the following conferences, where part of this research was conducted: "Random Functions, Random Surfaces and Interfaces" at CRM (January, 2009); "Random Fields and Stochastic Geometry" at Banff International Research Station (February, 2009). D.J. would like to also thank the organizers of the program "Selected topics in spectral theory" at Erwin Shrödinger Institute in Vienna (May 2009), as well as the organizers of the conference "Topological Complexity of Random Sets" at American Institute of Mathematics in Palo Alto (August 2009).

2. Random metrics in a conformal class

We consider a conformal class of metrics on a Riemannian manifold M of the form

$$g_1 = e^{af} g_0, \tag{1}$$

where g_0 is a "reference" Riemannian metric on M, a is a constant, and $f = f(x)$ is a C^2 function on M.

Given a metric g_0 on M and the corresponding Laplacian Δ_0, let $\{\lambda_j, \phi_j\}$ denote an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta_0$; we let $\lambda_0 = 0, \phi_0 = 1$. We define a random conformal multiple $f(x)$ by

$$f(x) = -\sum_{j=1}^\infty a_j c_j \phi_j(x), \tag{2}$$

where $a_j \sim \mathcal{N}(0,1)$ are i.i.d standard Gaussians, and c_j are positive real numbers, and we use the minus sign for convenience purposes only. We assume that $c_j = F(\lambda_j)$, where $F(t)$ is an eventually monotone decreasing function of t, $F(t) \to 0$ as $t \to \infty$. For example, we may take $c_j = e^{-\tau\lambda_j}$ or $c_j = \lambda_j^{-s}$. Equivalently we equip the space of functions (distributions) $L^2(M)$ with the probability measure $\nu = \nu_{\{c_n\}_{n=1}^\infty}$ generated by the densities on the finite cylinder sets

$$d\nu_{(n_1,n_2,\ldots n_l)}(f) = \frac{1}{\prod_{j=1}^{l} (2\pi c_{n_j}^2)^{1/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{l} \frac{f_{n_j}^2}{c_{n_j}^2}\right) df_{n_1} \ldots df_{n_l}, \tag{3}$$

where $f_n = \langle f, \phi_j \rangle_{L^2(M)}$ are the Fourier coefficients.

The *random field* $f(x)$ is a centered Gaussian field with covariance function

$$r_f(x,y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y),$$

$x, y \in M$. In particular for every $x \in M$, $f(x)$ is mean zero Gaussian of variance

$$\sigma^2(x) = r_f(x,x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2.$$

For special manifolds such as the 2-dimensional sphere $\mathcal{S}^2 \subseteq \mathbb{R}^3$ it will be convenient to parameterize $f(x)$ in a different fashion (see (24)).

The central object of the present study is the scalar curvature resulting from the conformal change of the metric (1). For $n = 2$, the expression (8) below for the curvature is of a particular simple shape, studying which it is convenient to work with the random centered Gaussian field

$$h(x) := \Delta_0 f(x) = \sum_{j=1}^{\infty} a_j c_j \lambda_j \phi_j(x) \tag{4}$$

having the covariance function

$$r_h(x,y) = \sum_{j=1}^{\infty} c_j^2 \lambda_j^2 \phi_j(x)\phi_j(y) \tag{5}$$

$x, y \in M$. In principle, one may derive any property of h in terms of the function r_h and its derivatives by the Kolmogorov theorem.

2.1. **Smoothness.** We refer to [Au98, Ch. 2] for definitions and basic facts about Sobolev spaces.

The smoothness of the Gaussian random field (2) is given by the following proposition, [Bl, Proposition 1]:

Proposition 2.1. *If* $\sum_{j=1}^{\infty} (\lambda_j + 1)^r c_j^2 < \infty$, *then* $f(x) \in H^r(M)$ *a.s. Equivalently, the measure* ν *defined as* (3) *is concentrated on* $H^r(M)$, *i.e.* $\nu(H^r) = 1$.

Choosing $c_j = F(\lambda_j) = \lambda_j^{-s}$ translates to $\sum_{j\geq 1} \lambda_j^{r-2s} < \infty$. In dimension n, it follows from Weyl's law that $\lambda_j \asymp j^{2/n}$ as $j \to \infty$; we find that

$$If\ s > \frac{2r+n}{4}, \quad then\ f(x) \in H^r(M)\ a.s.$$

By the Sobolev embedding theorem, $H^r \subset C^k$ for $k < r - n/2$. Substituting into the formula above, we find that

$$If\ c_j = O(\lambda_j^{-s}), s > \frac{n+k}{2}, \quad then\ f(x) \in C^k\ a.s. \tag{6}$$

We will be mainly interested in $k = 0$ and $k = 2$. Accordingly, we formulate the following

Corollary 2.2. *If* $c_j = O(\lambda_j^{-s}), s > n/2$, *then* $f \in C^0$ *a.s; if* $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, *then* $f \in C^2$ *a.s. Similarly, if* $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, *then* $\Delta_0 f \in C^0$ *a.s; if* $c_j = O(\lambda_j^{-s}), s > n/2 + 2$, *then* $\Delta_0 f \in C^2$ *a.s.*

2.2. **Volume.** We next consider the volume of the random metric in (1). The volume element dV_1 corresponding to g_1 is given by

$$dV_1 = e^{naf/2} dV_0, \tag{7}$$

where dV_0 denotes the volume element corresponding to g_0.

We consider the random variable $V_1 = \mathrm{vol}(M, g_1)$. We shall prove the following

Proposition 2.3. *Notation as above,*

$$\lim_{a\to 0} \mathbb{E}[V_1(a)] = V_0,$$

where V_0 denotes the volume of (M, g_0).

Proof. Recall that $f(x)$ defined by (2) is a mean zero Gaussian with variance $\sigma(x)^2 = r_f(x,x)$. One may compute explicitly

$$\mathbb{E}[e^{naf(x)/2}] = e^{\frac{1}{8}n^2a^2 r_f(x,x)},$$

so that (7) implies that

$$\mathbb{E}[dV_1(x)] = e^{\frac{1}{8}n^2a^2 r_f(x,x)} dV_0(x).$$

Hence, using Fubini we obtain

$$\mathbb{E}[V_1(a)] = \int_M \mathbb{E}[dV_1] = \int_M e^{\frac{1}{8}n^2a^2 r_f(x,x)} dV_0.$$

Since $r_f(x,x)$ is continuous, as $a \to 0$, the latter converge to V_0 by the dominated convergence theorem (say). □

Remark 2.4. *The smoothness of the metric $g_1 = e^{af} g_0$ is almost surely determined by the coefficients c_j. In some sense, a can be regarded as the radius of a sphere (in an appropriate space of Riemannian metrics on M) centered at g_0. Most of the results in this paper hold in the limit $a \to 0$; thus, we are studying* local *geometry of the space of Riemannian metrics on M.*

2.3. **Scalar curvature in a conformal class.** It is well-known that the scalar curvature R_1 of the metric g_1 in (1) is related to the scalar curvature R_0 of the metric g_0 by the following formula ([Au98, §5.2, p. 146])

$$R_1 = e^{-af}[R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4], \tag{8}$$

where Δ_0 is the (negative definite) Laplacian for g_0, and ∇_0 is the gradient corresponding to g_0. We observe that the last term vanishes when $n = 2$:

$$R_1 = e^{-af}[R_0 - a\Delta_0 f]. \tag{9}$$

Substituting (2), we find that

$$R_1(x)e^{af(x)} = R_0(x) - a\sum_{j=1}^{\infty} \lambda_j a_j c_j \phi_j(x). \tag{10}$$

The smoothness of the scalar curvature for the metric g_1 is determined by the random field $a(n-1)\Delta f + a^2(n-1)(n-2)|\nabla f|^2/4$.

We remark that it follows easily from (8) and Corollary 2.2 that

Proposition 2.5. *If $R_0 \in C^0$ and $c_j = O(\lambda_j^{-s}), s > n/2+1$ then $R_1 \in C^0$ a.s. If $R_0 \in C^2$ and $c_j = O(\lambda_j^{-s}), s > n/2+2$ then $R_1 \in C^2$ a.s.*

Consider the *sign* of the scalar curvature R_1 of the new metric. We make a remark that will be important later:

Remark 2.6. *Note that the quantity e^{-af} is positive so that the sign of R_1 satisfies*

$$\operatorname{sgn}(R_1) = \operatorname{sgn}[R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4],$$

in particular for $n = 2$, assuming that R_0 has constant sign, we find that

$$\operatorname{sgn}(R_1) = \operatorname{sgn}(R_0 - a\Delta_0 f) = \operatorname{sgn}(R_0 - ah) = \operatorname{sgn}(R_0) \cdot \operatorname{sgn}(1 - ah/R_0).$$

We shall later study similar questions for Branson's Q-curvature [BG, DM, N].

3. Using Borel-TIS inequality to estimate the probability that R_1 changes sign

In this section, we shall use Borell-TIS inequality 3.1 to estimate the probability that curvature of a random metric on a compact orientable surface M of genus $\gamma \neq 1$ changes sign. We remark that by Gauss-Bonnet theorem, for $M = \mathbf{T}^2$ we have $\int_M R = 0$, so the curvature has to change sign on $\mathbf{T}^2$ (while for flat metrics, $R_0 \equiv 0$).

We denote by $M = M_\gamma$ a compact surface of genus $\gamma \neq 1$. We choose a reference metric g_0 so that R_0 has constant sign (positive iF $M = S^2$, and negative if M has genus 2). We remark that by uniformization theorem, such metrics exist in every conformal class. In fact, every metric on M is conformally equivalent to a metric with $R_0 \equiv$ *const*.

Define the random metric on M_γ by $g_1 = e^{af} g_0$, (as in (1)) and f is given by (2), as usually.

In this section we shall estimate the probability $P_2(a)$ defined by

$$P_2(a) := \operatorname{Prob}\{\exists x \in M : \operatorname{sgn} R_1(g_1(a), x) \neq \operatorname{sgn}(R_0)\}, \tag{11}$$

i.e. that the curvature R_1 of the random metric $g_1(a)$ *changes sign* somewhere on M. The probability of the complementary event $P_1(a) = 1 - P_2(a)$ is clearly

$$P_1(a) := \operatorname{Prob}\{\forall x \in M : \operatorname{sgn}(R_1(g_1(a), x)) = \operatorname{sgn}(R_0), \}$$

i.e. the curvature of the random metric $g_1(a)$ *does not* change sign.

Recall by Remark 2.6, in dimension two $\operatorname{sgn}(R_1) = \operatorname{sgn}(R_0) \operatorname{sgn}(1 - ah/R_0)$, where $h = \Delta_0 f$ was defined earlier in (4). We let v denote the random field

$$v(x) = h(x)/R_0(x) \tag{12}$$

We remark that

$$r_v(x, x) = r_h(x, x)/[R_0(x)]^2, \tag{13}$$

and we let

$$\sigma_v^2 = \sup_{x \in M} r_v(x, x) = \sup_{x \in M} r_h(x, x)/[R_0(x)]^2. \tag{14}$$

We denote by $||v||_M := \sup_{x \in M} v(x)$. It follows from Remark 2.6 that

$$P_2(a) = \operatorname{Prob}\{||v||_M > 1/a\} \tag{15}$$

We shall estimate $P_2(a)$ in the limit $a \to 0$. Geometrically, that means that $g_1(a) \to g_0$, so $P_2(a)$ should go to zero as $a \to 0$; below, we shall estimate the *rate*. To do that, we shall use a strong version of the Borell-TIS inequality ([Bor, TIS]) formulated below.

The proof of the following result can be found in [Bor, TIS], or in [AT08, p. 51]

Theorem 3.1 (Borel-TIS). *Let f be a centered Gaussian process, a.s. bounded on M, and $\sigma_M^2 := \sup_{x\in M} \mathbb{E}[f(x)^2]$. Then $\mathbb{E}\{||f||_M\} < \infty$, and there exists a constant α depending only on $\mathbb{E}\{||f||_M\}$ so that for $u > E\{||f||_M\}$ we have*

$$\mathrm{Prob}\{||f||_M > u\} \leq e^{\alpha u - u^2/(2\sigma_M^2)}.$$

From now on we shall assume that $R_0 \in C^0(M)$, and that $c_j = O(\lambda_j^{-s}), s > 2$. Then Proposition 2.5 implies that h and R_1 are a.s. C^0 and hence bounded, since M is compact.

Recall that $h(x) := \sum_{j=1}^{\infty} \lambda_j c_j a_j \phi_j(x)$; it follows that the variance of $v = h/R_0$ is equal to $(\sum_{j=1}^{\infty} c_j^2 \lambda_j^2 \phi_j(x)^2)/(R_0(x)^2)$.

Recall (14) that $\sigma_v^2 = \sup_{x\in M} r_v(x,x)$; assume that the supremum is attained at $x = x_0$. We shall use (15) to estimate $P_2(a)$ from above and below. To get a lower bound for $\mathrm{Prob}\{\|v\|_M\} > 1/a)$, choose $x = x_0$. Clearly,

$$\mathrm{Prob}\{\|v\|_M\} > 1/a\} \geq \mathrm{Prob}\{v(x_0) > 1/a\}.$$

The random variable $v(x_0)$ is Gaussian with mean 0 and variance σ_v^2. Accordingly,

$$\mathrm{Prob}\{v(x_0) > 1/a\} = \Psi\left(\frac{1}{a\sigma_v}\right), \tag{16}$$

where we denote the error function

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^{\infty} e^{-t^2/2} dt.$$

We obtain an upper bound by a straightforward application of Theorem 3.1 on our problem.

Proposition 3.2. *There exist a constant C so that*

$$\mathrm{Prob}\{\|v\|_M) > 1/a\} \leq e^{C/a - 1/(2a^2\sigma_v^2)}.$$

Combining Proposition 3.2 and (16) we obtain the following theorem. Note that the coefficient of a^{-2} in the exponent is the same on both sides.

Theorem 3.3. *Assume that $R_0 \in C^0(M)$ and that $c_j = O(\lambda_j^{-s}), s > 2$. Then there exist constants $C_1 > 0$ and C_2 such that the probability $P_2(a)$ satisfies*

$$(C_1 a) e^{-1/(2a^2\sigma_v^2)} \leq P_2(a) \leq e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as $a \to 0$. In particular

$$\lim_{a\to 0} a^2 \ln P_2(a) = \frac{-1}{2\sigma_v^2}.$$

Remark 3.4. *In section 4, we shall greatly improve the result of Theorem 3.3 and obtain much more precise estimates of $P_2(a)$ for $M = \mathcal{S}^2$ (see Theorem 4.3 below) using the results of Adler and Taylor described in the next section. To apply Borell-TIS inequality, h is required to be a.s. C^0. To apply the results of Adler-Taylor, h needs to be a.s. C^2. We hope to improve the estimates in Theorem 3.3 in a forthcoming paper.*

3.1. **Random real-analytic metrics and comparison results.** In this section, we let M be a compact orientable surface, $M \ncong \mathbf{T}^2$. We shall consider random *real-analytic* conformal deformations; this corresponds to the case when the coefficients c_j in (2) decay *exponentially.* We shall use standard estimates for the heat kernel to estimate the probabilities computed in the previous section 3.

We fix a real parameter $T > 0$ and choose the coefficients c_j in (2) to be equal to

$$c_j = e^{-\lambda_j T/2}/\lambda_j. \tag{17}$$

Then it follows from (5) that

$$r_h(x,x) = e^*(x,x,T) = \sum_{j:\lambda_j>0} e^{-\lambda_j T}\phi_j(x)^2,$$

where $e^*(x,x,T)$ denotes the *heat kernel* on M *without the constant term*, evaluated at x at time T.

The heat kernel $e(x,y,t) = \sum_j e^{-\lambda_j t}\phi_j(x)\phi_j(y)$ defines a fundamental solution of the heat equation on M. It is well-known that $e(x,y,t)$ is smooth in x,y,t for $t > 0$, and that $e^*(x,y,t)$ decays exponentially in t, [Ch, Gil].

3.2. **Comparison Theorem:** $T \to 0^+$**.** The following asymptotic expansion for the heat kernel is standard [Gil]:

$$e(x,x,T) \sim_{t\to 0^+} \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) T^{j-n/2};$$

here $a_j(x)$ is the j-th *heat invariant*, where

$$a_0(x) = 1,\ a_1(x) = R(x)/6.$$

In particular,

$$\lim_{T\to 0^+} e(x,x,T)T^{n/2} = \frac{1}{(4\pi)^{n/2}}.$$

Combining with (14), we obtain the following

Proposition 3.5. *Assume that the coefficients c_j are chosen as in* (17)*. Then as $T \to 0^+$, σ_v^2 is asymptotic to*

$$\frac{1}{(4\pi T)^{n/2} \inf_{x\in M}(R_0(x))^2}.$$

That is, as $T \to 0^+$, the probability $P_2(a)$ is determined by the value of

$$\inf_{x\in M}(R_0(x))^2.$$

Proposition 3.5 is next applied to prove a comparison theorem. Let g_0 and g_1 be two distinct reference metrics on M, normalized to have equal volume, and such that $R_0 \equiv const$ and $R_1 \not\equiv const$.

Theorem 3.6. *Let g_0 and g_1 be two distinct reference metrics on M, normalized to have equal volume, such that R_0 and R_1 have constant sign, $R_0 \equiv const$ and $R_1 \not\equiv const$. Then there exists $a_0, T_0 > 0$ (that depend on g_0, g_1) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $P_2(a,T,g_1) > P_2(a,T,g_0)$.*

Proof: It follows from Gauss-Bonnet's theorem that

$$\int_M R_0 dV_0 = \int_M R_1 dV_1.$$

Since $\mathrm{vol}(M, g_0) = \mathrm{vol}(M, g_1)$, and since by assumption $R_0 \equiv const$ and $R_1 \not\equiv const$, it follows that

$$b_0 := \min_{x\in M}(R_0(x))^2 > \inf_{x\in M}(R_1(x))^2 := b_1.$$

Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma_v^2(g_1, T)}{\sigma_v^2(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$

The result now follows from Theorem 3.3.

□

It follows that in every conformal class, $P_2(a, T, g_0)$ is minimized in the limit $a \to 0, T \to 0^+$ for the metric g_0 of constant curvature.

3.3. **Comparison Theorem: $T \to \infty$.** Let M be a compact surface, where the scalar curvature R_0 of the reference metric g_0 has constant sign. Let $\lambda_1 = \lambda_1(g_0)$ denote the smallest nonzero eigenvalue of Δ_0. Denote by $m = m(\lambda_1)$ the multiplicity of λ_1, and let

$$F := \sup_{x\in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{R_0(x)^2}. \tag{18}$$

The number F is finite by compactness and the assumption that R_0 has constant sign on M.

Proposition 3.7. *Let the coefficients c_j be as in* (17)*. Denote by $\sigma_v^2(T)$ the corresponding supremum of the variance of v. Then*

$$\lim_{T\to\infty} \frac{\sigma_v^2(T)}{Fe^{-\lambda_1 T}} = 1. \tag{19}$$

Proof of Proposition 3.7: Recall that it follows from (14) that

$$r_v(x, x) = \frac{e^*(x, x, T)}{R_0(x)^2}.$$

We write $e^*(x, x, T) = e_1(x, T) + e_2(x, T)$, where

$$e_1(x, T) = e^{-\lambda_1 T} \sum_{j=1}^m \phi_j(x)^2,$$

and

$$e_2(x, T) = \sum_{j=m+1}^\infty e^{-\lambda_j T} \phi_j(x)^2.$$

Clearly, as $T \to \infty$, we have

$$\lim_{T\to\infty} e^{\lambda_1 T} \sup_{x\in M} \frac{e_1(x, T)}{R_0(x)^2} = F,$$

where F was defined in (18). It suffices to show that as $T \to \infty$,

$$\frac{e_2(x, T)}{R_0(x)^2} = o\left(e^{-\lambda_1 T}\right) \tag{20}$$

Note that by compactness, there exists $C_1 > 0$ such that $(1/C_1) \le R_0^2(x) \le C_1$ for all $x \in M$. Accordingly, it suffices to establish (20) for $\sup_{x\in M} e_2(x,T)$.

We let $\mu := \lambda_{m+1} - \lambda_m$; note that $\lambda_m = \lambda_1$ by the definition of m. We have

$$e_2(x,T) = e^{-\lambda_1 T} \sum_{j=m+1}^{\infty} e^{-(\lambda_j-\lambda_1)T}\phi_j(x)^2. \tag{21}$$

Let k be the smallest number such that $\lambda_k > 2\lambda_1$.

We rewrite the sum in (21) as $e_2 = e_3 + e_4$, where the first term e_3 is given by

$$e_3(x,T) := \sum_{j=m+1}^{k-1} e^{-(\lambda_j-\lambda_1)T}\phi_j(x)^2 \le e^{-\mu T} \sup_{x\in M} \sum_{j=m+1}^{k-1} \phi_j(x)^2, \tag{22}$$

where the last supremum (which we denote by H) is finite by compactness.

The second term e_4 is given by

$$e_4(x,T) := \sum_{j=k}^{\infty} e^{-(\lambda_j-\lambda_1)T}\phi_j(x)^2 \le \sum_{j=k}^{\infty} e^{-\lambda_j T/2}\phi_j(x)^2 \le \sup_{x\in M} e^*(x,x,T/2). \tag{23}$$

We remark that as $T \to \infty$, $e^*(x,x,T/2) \to 0$ exponentially fast, uniformly in x.

Combining (22) and (23), we find that

$$e_2(x,T) = O\left(\left[H \cdot e^{-\mu T} + e^*(x,x,T/2)\right] e^{-\lambda_1 T}\right),$$

establishing (20) for $\sup_{x\in M} e_2(x,T)$ and finishing the proof of Proposition 3.7.

□

Theorem 3.8. *Let g_0 and g_1 be two reference metrics (of equal area) on a compact surface M, such that R_0 and R_1 have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on g_0, g_1), such that for all $a < a_0$ and $T > T_0$ we have $P_2(a,T;g_0) < P_2(a,T;g_1)$.*

Proof of Theorem 3.8: By Proposition 3.7, we find that for $T > T_1 = T_1(g_0, g_1)$ there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\sigma_v^2(T,g_1)e^{\lambda_1(g_1)T}}{\sigma_v^2(T,g_2)e^{\lambda_1(g_2)T}} \le C$$

Accordingly, if we choose T_2 so that $e^{(\lambda_1(g_1)-\lambda_1(g_2))T_2} > C$, and take $T > \max\{T_1, T_2\}$, we find that Theorem 3.8 follows from the formula above and Theorem 3.3.

□

It was proved by Hersch in [Her] that for $M = S^2$, if we denote by g_0 the round metric on S^2, then $\lambda_1(g_0) > \lambda_1(g_1)$ for any other metric g_1 on S^2 of equal area. This immediately implies the following

Corollary 3.9. *Let g_0 be the round metric on S^2, and let g_1 be any other metric of equal area. Then, there exist $a_0 > 0$ and $T_0 > 0$ (depending on g_1) such that for all $a < a_0$ and $T > T_0$ we have $P_2(a,T;g_0) < P_2(a,T;g_1)$.*

It seems interesting to establish comparison results for finite times $0 < T < \infty$. In fact, it was proved in [Mor] that the heat trace for the round metric on S^2 locally minimizes the heat trace for all metrics on S^2 of the same volume, in an L^∞ neighborhood of the set of conformal factors on S^2; the size of the neighborhood depends on the interval $[a,b] \subset (0,\infty)$, where $T \in [a,b]$. It was also shown in [EI02], that the round metric on S^2 was the unique critical metric on S^2 for heat trace functional. Accordingly, it seems natural to conjecture that the round metric on S^2 will be extremal for $P_2(a,T)$ for all T, in the limit $a \to 0$.

For surfaces of genus $\gamma \geq 2$ the situation is different. The following result was proved in [Bryant, Theorem 2.3] (D.J. first learned about it from S. Wolpert); the corresponding result for minimal surfaces in $\mathbf{R}^n$ was established in [Yau74, Thm. 6].

Proposition 3.10. *Let g_0 be a hyperbolic metric on a compact orientable surface M of genus $\gamma \geq 2$. Then g_0* does not *maximize λ_1 in its conformal class.*

It is well-known that a metric g_0 that is extremal for λ_k among all metrics of the same volume in the same conformal class admits a *minimal immersion into S^m* by eigenfunctions that form an orthonormal basis in the eigenspace $E(\lambda_k)$, where $\dim E(\lambda_k) = m + 1$, cf. [EI03, EI08, Nad, Tak]. Strong results about the existence of such metrics were established recently in [NS].

The metrics that are extremal for λ_k among *all* metrics of the same volume (and not just in the same conformal class) admit *isometric* minimal immersions into round spheres by the corresponding eigenfunctions, see the references above, as well as [Her, LY, YY]. A metric that maximizes λ_1 for surfaces of genus 2 is a branched covering of the round 2-sphere, cf. [JLNNP].

Accordingly, we conclude that on surfaces of genus $\gamma \geq 2$, different metrics maximize $P_2(a,T)$ in the limit $a \to 0, T \to 0$ and in the limit $a \to 0, T \to \infty$, unlike the situation on S^2.

4. Using results of [AT08] on the 2-Sphere

The sphere is special in that the curvature perturbation is *isotropic*, so that in particular the variance is constant. In this case a special theorem due to Adler-Taylor gives a precise asymptotics for the excursion probability.

4.1. **Random function in Sobolev spaces.** For an integer m let $\mathcal{E}_m$ be the space of spherical harmonics of degree m of dimension $N_m = 2m + 1$ associated to the eigenvalue $E_m = m(m + 1)$, and for every m fix an L^2 orthonormal basis $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ of $\mathcal{E}_m$.

To treat the spectrum degeneracy it will be convenient to use a slightly different parametrization of the conformal factor than the usual one (2)

$$f(x) = -\sqrt{|\mathcal{S}^2|} \sum_{m\geq 1,\, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x), \tag{24}$$

where $a_{m,k}$ are standard Gaussian i.i.d. and $c_m > 0$ are some (suitably decaying) constants. For extra convenience we will assume in addition that

$$\sum_{m=1}^{\infty} c_m = 1, \tag{25}$$

which has an advantage that a random field $h(x)$ defined below is of unit variance.

Remark 4.1. *We stress once again that for convenience, in the present section, the random fields f and h (see below) are defined differently than in the rest of the paper. The reason for the new definitions is spectral degeneracy on S^2.*

The measure $\nu = \nu_{\{c_m\}_{m=1}^{\infty}}$ corresponding to (3) is generated by the densities on the finite cylinder sets

$$d\nu_{(m_1,k_1),\dots(m_l,k_l)}(f) = \frac{1}{\prod\limits_{i=1}^{l}(2\pi s_{m_i})^{1/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{l}\frac{f^2_{(m_i,k_i)}}{s_{m_i}}\right) df_{(m_1,k_1)}\dots df_{(m_l,k_l)},$$

where $f_{(m,k)} = \langle f, \eta_{m,k}\rangle_{L^2(\mathcal{S}^2)}$ are the Fourier coefficients, and

$$s_m := |\mathcal{S}^2|\frac{c_m}{E_m^2 N_m}.$$

Note that ν is invariant w.r.t. the choice of the orthonormal basis $\{B_m\}_{m=1}^{\infty}$ of the spaces $\mathcal{E}_m$ of the spherical harmonics, by the invariance of the Gaussian.

Recall that the Sobolev space $H_r(\mathcal{S}^2)$ consists of functions (distributions) $g : \mathcal{S}^2 \to \mathbb{R}$, so that

$$\sum_{m,k}(E_m+1)^r g_{m,k}^2 < \infty.$$

For example $L^2(\mathcal{S}^2) = H_0(\mathcal{S}^2)$.

By [Bl], Proposition 1, ν is concentrated on $H_r(\mathcal{S}^2)$, if and only if

$$\sum_{m=1}^{\infty} N_m(E_m+1)^r\frac{c_m}{E_m^2 N_m} < \infty.$$

Since

$$(E_m+1)^r\frac{c_m}{E_m^2} \asymp m^{2r-4}c_m,$$

we have the following lemma.

Lemma 4.2. *Given a sequence c_m satisfying* (25)*, we have $f(x) \in H_r(\mathcal{S}^2)$ a.s. (or equivalently, the measure ν defined above satisfies $\nu(H_r) = 1$) if and only if*

$$\sum_{m=1}^{\infty} m^{2r-4}c_m < \infty.$$

In what follows we will always assume that

$$c_m = O\left(\frac{1}{m^s}\right). \tag{26}$$

Thus $f(x) \in H_r(\mathcal{S}^2)$ precisely for $r < \frac{s}{2} + \frac{3}{2}$. Note that if $c_m = \frac{K}{m^s}$, (25) requires $K = \frac{1}{\zeta(s)}$ where $\zeta(s)$ is the Riemann zeta function.

4.2. **Curvature and statement of the main result.**

Theorem 4.3. *Let $s > 7$, and the metric g_1 on $\mathcal{S}^2$ be given by*

$$g_1 = e^{af}g_0$$

where f is given by (24)*. Also, let $c_m \neq 0$ for at least one* odd *m. Then as $a \to 0$, the probability that the curvature is everywhere positive is given by*

$$\begin{aligned}\text{Prob}\{\forall x \in \mathcal{S}^2.\, R_1(x) > 0\} &= 1 - C_1\Psi\left(\frac{1}{a}\right) - \frac{C_2}{a}\exp\left(-\frac{1}{2a^2}\right) + o\left(\exp(-\frac{\alpha}{2a^2})\right)\\ &\sim 1 - C_1\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2a^2}\right) - \frac{C_2}{a}\exp\left(-\frac{1}{2a^2}\right),\end{aligned}$$

where $C_1 = 2$, $C_2 = \frac{1}{\sqrt{2\pi}}\sum_{m\geq 1} c_m E_m$ and $\alpha > 1$.

The curvature corresponding to the random Riemannian metric $g_1 = e^{af} g_0$ is given by

$$R_1 e^{af} = R_0 - a\Delta f = 1 - a\Delta f, \tag{27}$$

where $R_0 \equiv 1$ corresponds to the round metric g_0. To make sense of it we will have to assume that $f \in C^2(\mathcal{S}^2)$ a.s., for which we will need that $s > 3$ (cf. (26))

It is then natural to introduce the Gaussian random field (cf. (4))

$$h(x) := \Delta f(x) = \sqrt{|\mathcal{S}^2|} \sum_{m\geq 1,\, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k} \eta_{m,k}(x), \tag{28}$$

so that (27) is

$$R_1 e^{af} = 1 - ah. \tag{29}$$

The random field h is centered unit variance (see (25)) Gaussian isotropic with covariance function $r_h : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathbb{R}$ given explicitly by

$$r_h(x,y) := \mathbb{E}[h(x)h(y)] = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x,y))), \tag{30}$$

where P_m is the Legendre polynomial of degree m and $d(x,y)$ is the (spherical) distance between x and y.

The following lemma follows easily from (28), Proposition 2.1 and Sobolev embedding theorem:

Lemma 4.4. *If $c_m = O(m^{-s})$, $s > 2k+3$, then h, and hence R_1, are a.s. C^k.*

The condition (26) in particular ensures that the series in (28) is a.s. pointwise convergent; we will need stronger conditions to work with *smooth* sample functions. It then follows from (29) that R_1 is everywhere positive if and only if

$$\|h\|_{\mathcal{S}^2} := \sup_{x\in\mathcal{S}^2} \{h(x)\} < \frac{1}{a}.$$

The problem of approximating the excursion probability of

$$E = E_{h,u} := \left\{ \|h\|_{\mathcal{S}^2} > u := \frac{1}{a} \right\}$$

(i.e., of the complement event) for a given random field h as $u \to \infty$ (i.e. $a \to 0$, small perturbation) is a classical problem in probability. For the constant variance random fields (which follows from the isotropic property of h), there is a special precise result due to Adler-Taylor [AT03]. The latter relates Prob(E) to the expected value of Euler characteristic of the excursion set $h^{-1}([u,\infty])$, giving an explicit expression for the latter, where the answer depends on the Adler-Taylor metric associated to h defined below.

4.3. **The Adler-Taylor metric of h.** Let h be an a.s. C^1 random field on a manifold M. The *Adler-Taylor* Riemannian metric g^{AT} on M is defined as follows (cf. [AT08, (12.2.2)]). Let $x \in M$ and $X, Y \in T_x M$; then

$$g^{AT}_{h;x}(X,Y) := \mathbb{E}[Xh \cdot Yh].$$

This is the pullback by $x \to h(x)$ of the standard structure on L^2. One may compute g^{AT} in terms of the covariance function as ([AT08], p. 306)

$$g^{AT}_{h;x}(X,Y) = XY r_h(x,y)|_{x=y},$$

Below, we shall specialize to the case $M = \mathcal{S}^2$.

Plugging in (30) we obtain an expression for the metric

$$g_{h;x}^{AT}(X,Y) = \sum_{m=1}^{\infty} c_m \left(XY P_m(\cos(d(x,y))) \right)|_{x=y},$$

which was computed explicitly for the sphere in [W, W1] to be given by the scalar matrix

$$g_{h;x}^{AT} = CI_2 \tag{31}$$

with $C = C_{c_j} := \frac{1}{2}\sum_{m=1}^{\infty} c_m E_m$, in any orthonormal basis of $T_x(\mathcal{S}^2)$.

Definition 4.5. *Let M be a smooth manifold and $f : M \to \mathbb{R}$ a smooth centered Gaussian random field with covariance function $r_f(x,y)$. We say that f is* attainable, *if there exists a countable atlas $\mathcal{A} = (U_\alpha, \psi_\alpha)_{\alpha \in I}$ on M, such that for every $\alpha \in I$, $f^\alpha := f \circ \psi_\alpha^{-1}$ defined on $\psi(U_\alpha) \subseteq \mathbb{R}^2$, satisfies:*

(1) *For each $t \in \psi(U_\alpha)$, the joint distributions of*

$$(f_i^\alpha(t), f_{ij}^\alpha(t))_{i<j} \in \mathbb{R}^5$$

are nondegenerate, where f_i^α and f_{ij}^α are the corresponding partial derivatives of f^α of first and second order respectively.

(2) *We have (cf.* [AT08]*, (11.3.1))*

$$\max_{i,j} \left| r_{h_{ij}}(t,t) + r_{h_{ij}}(s,s) - 2r_{h_{ij}}(s,t) \right| \leq K_\alpha [\ln |t-s|]^{-(1+\beta)}$$

for some $\beta > 0$.

Theorem 4.6 ([AT08], Theorem 12.4.1, 2-dimensional case)**.** *Let $f : M \to \mathbb{R}$ be a centered, unit variance Gaussian field on a C^2, 2-dimensional manifold M. Then if f is attainable,*

$$\mathbb{E}[\chi(u, +\infty)] = \sum_{j=0}^{2} \mathcal{L}_j(M)\rho_j(u),$$

where $\mathcal{L}_j(\mathcal{S}^2)$, $j = 0, 1, 2$, are the Lipschitz-Killing curvatures of M computed w.r.t. the Adler-Taylor metric $g_{f,\cdot}^{AT}$ and

$$\rho_j(u) = \begin{cases} \Psi(u) & j = 0, \\ \frac{1}{(2\pi)^{1/2}} e^{-u^2/2} & j = 1, \\ \frac{1}{(2\pi)^{3/2}} u e^{-u^2/2} & j = 2. \end{cases}$$

We will apply Theorem 4.6 on the random field $h = \Delta f$ in the course of proof of the main result of this section, namely Theorem 4.3; we relegate the justification of h being attainable to Appendix C.1.

Theorem 4.6 allows us to compute the expected Euler characteristic of the excursion set, which is intimately related to the excursion probability by [AT08], (14.0.2):

$$\left| \text{Prob}\{\|h\|_{\mathcal{S}^2} \geq u\} - \mathbb{E}[\chi\left(h^{-1}[u, \infty)\right)] \right| = O\left(e^{-\alpha u^2/2}\right) \tag{32}$$

for some $\alpha > 1$; this is a meta-theorem (here we used assumption (25); otherwise we need to modify accordingly). Theorem 14.3.3 in [AT08] gives sufficient conditions for this assertions to hold; we relegate validating its hypotheses to Appendix C.2.

4.4. **Proof of Theorem 4.3.**

Proof of Theorem 4.3. We are interested in the probability that for every $x \in \mathcal{S}^2$

$$h(x) \leq u := \frac{1}{a}$$

for small $a > 0$, or, equivalently, its complement

$$\text{Prob}\{\|h\|_{\mathcal{S}^2} \geq u\}.$$

We employ Theorem 4.6 due to Adler-Taylor to compute the expected value of Euler characteristic of the excursion set explicitly as

$$\mathbb{E}[\chi\left(h^{-1}[u,\infty)\right)] = \sum_{j=0}^{2} \mathcal{L}_j(\mathcal{S}^2)\rho_j(u), \tag{33}$$

The statement of Theorem 4.3 then follows from (32), (33), and the values of the Lipschitz-killing curvatures of the sphere relatively to the Adler-Taylor metric (31) $\mathcal{L}_0(\mathcal{S}^2, g_h^{AT}) = 2$, $\mathcal{L}_1(\mathcal{S}^2, g_h^{AT}) = 0$ and $\mathcal{L}_2(\mathcal{S}^2, g_h^{AT}) = 2\pi\left(\sum_{m\geq 1}\frac{c_m E_m}{2}\right)$ (see [AT08], (6.3.8)). Note that to justify the application of Theorem 4.3 and (32) we have to validate the hypotheses of the corresponding theorems. We do so in Appendix C.

□

5. L^∞ CURVATURE BOUNDS

5.1. **Definitions and the main result.** In sections 3 and 4 we studied the probability of the curvature *changing sign* after a small conformal perturbation, on S^2 and on surfaces of genus greater than one. On the torus $\mathbf{T}^2$, however, Gauss-Bonnet theorem implies that the curvature has to change sign for every metric, so that question is meaningless.

Accordingly, on $\mathbf{T}^2$ we investigate the probability of another event that is considered very frequently in comparison geometry: the probability that scalar curvature satisfies the L^∞ curvature bounds $||R_1||_\infty < u$, where $u > 0$ is a parameter. Metrics satisfying such bounds for fixed u are called *metrics of bounded geometry.* The argument on $\mathbf{T}^2$ is then modified to study the following natural analogue of the problem on $\mathbf{T}^2$: estimating the probability that $||R_1 - R_0||_\infty < u$. That question is considered on S^2, and on surfaces of genus greater than one.

In this section we *do not* assume that $R_0 \equiv const$; nor do we assume that R_0 has constant sign.

Definition 5.1. *We shall consider the following three centered random fields on the surface Σ:*

i) *The random conformal multiple $f(x)$ given by (2). We denote its covariance function by $r_f(x,y)$, and we define $\sigma_f^2 = \sup_{x\in\Sigma} r_f(x,x)$.*
ii) *The random field $h = \Delta_0 f$ defined in (4). We denote its covariance function by $r_h(x,y)$, and we define $\sigma_h^2 = \sup_{x\in\Sigma} r_h(x,x)$.*
iii) *The random field $w = \Delta_0 f + R_0 f = h + R_0 f$. We denote its covariance function by $r_w(x,y)$, and we define $\sigma_w^2 = \sup_{x\in\Sigma} r_w(x,x)$. Note that on flat $\mathbf{T}^2$, $R_0 \equiv 0$ and therefore $h \equiv w$.*

The random fields f, h and w have constant variance on round S^2; also f and $h = w$ have constant variance on flat $\mathbf{T}^2$.

We shall prove the following theorem:

Theorem 5.2. *Assume that the random metric is chosen so that the random fields f, h, w are a.s. C^0. Let $a \to 0$ and $u \to 0$ so that*

$$\frac{u}{a} \to \infty. \tag{34}$$

Then

$$\log \mathrm{Prob}\{\|R_1 - R_0\|_\infty > u\} \sim -\frac{u^2}{2a^2\sigma_w^2}. \tag{35}$$

Remark 5.3. *On the flat 2-torus, $\sigma_h = \sigma_w$.*

5.2. **Proof of Theorem 5.2.**

Proof. In the proof, we shall use (Borel-TIS) Theorem 3.1; we require that the random fields f, h, w are a.s. C^0. Assuming that $R_0 \in C^0(\Sigma)$, sufficient conditions for that are formulated in Corollary 2.2; we remark that if h is a.s. C^0, then so is w.

Let Σ will denote a compact orientable surface ($\mathcal{S}^2, \mathbf{T}^2$ or of genus $\gamma \geq 2$) where the random fields are defined.

Step 1.

We introduce a (large) parameter S that will be chosen later. On $\mathbf{T}^2$, we let B_S denote the "bad" event where f is *large*

$$B_S = \{||f||_\infty > S\}. \tag{36}$$

Applying Theorem 3.1, we find that there exists a constant α_f such that the following estimate holds:

$$\mathrm{Prob}(B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right)\right). \tag{37}$$

On S^2 and on surfaces of genus ≥ 2 we modify the definition slightly, and let B_S denote the "bad" event that either f or h is large

$$B_S = \{||f||_\infty > S\} \cup \{||h||_\infty > S\}. \tag{38}$$

By Theorem 3.1 we find that there exist two constants α_f and α_h such that

$$\mathrm{Prob}(B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right). \tag{39}$$

We denote $A_{u,a}$ the event $\{||R_1 - R_0||_\infty > u\}$; clearly,

$$\mathrm{Prob}(A_{u,a}) = \mathrm{Prob}(A_{u,a} \cap B_S) + \mathrm{Prob}(A_{u,a} \cap B_S^c). \tag{40}$$

We will choose S later so that

$$\mathrm{Prob}(A_{u,a} \cap B_S) = o\left(\mathrm{Prob}(A_{u,a} \cap B_S^c)\right); \tag{41}$$

this is only possible under the assumption (34) of the present theorem. The inequality (41) implies that it will be sufficient to evaluate $\mathrm{Prob}(A_{u,a} \cap B_S^c)$.

Consider first the event $A_{u,a} \cap B_S^c$. The expression $R_1 - R_0$ is given by (43). We estimate $\mathrm{Prob}(A_{u,a} \cap B_S)$ trivially,

$$\mathrm{Prob}(A_{u,a} \cap B_S) \leq \mathrm{Prob}(B_S).$$

Accordingly, it follows from (37) for the torus, and (39) for the sphere or a surface of genus ≥ 2 that in each of the cases

(42)
$$\operatorname{Prob}(A_{u,a} \cap B_S) = \begin{cases} O\left(S \cdot \exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right)\right), & \Sigma = \mathbf{T}^2; \\ O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right), & \text{otherwise} \end{cases}.$$

Step 2.

We next estimate $\operatorname{Prob}(A_{u,a} \cap B_S^c)$. Recall that in dimension two, it follows from (9) that

$$R_1 - R_0 = R_0(e^{-af} - 1) - ae^{-af}\Delta_0 f = R_0(e^{-af} - 1) - ae^{-af}h. \tag{43}$$

Note that on $\mathbf{T}^2$ we have $R_0 = 0$, and the first term on the right vanishes, hence we get

$$R_1 = -ae^{-af}h$$

in that case.

Step 2a.

We start with the case $\Sigma = \mathbf{T}^2$. We will choose a constant S satisfying

$$aS = o(1). \tag{44}$$

On B_S^c, we have $|f(x)| = O(S)$, hence $e^{-af(x)} = 1 + O(aS)$, so that

$$\begin{aligned} \operatorname{Prob}(A_{u,a} \cap B_S^c) &= \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\} \cap B_S^c\right) \\ &= \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right) + O\left(\operatorname{Prob}(B_S)\right), \end{aligned} \tag{45}$$

the last summand being already estimated in (42). By (44), we have $\frac{u}{a(1+O(aS))} \sim \frac{u}{a}$. Plugging (42) and (45) into (40) we obtain

$$\operatorname{Prob}(A_{u,a}) = \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right) + O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right)\right). \tag{46}$$

It then remains to evaluate

$$\operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right),$$

and choose S so that the other term is negligible. To this end we note that by symmetry,

$$\begin{aligned} \operatorname{Prob}\left(\left\{\|h\|_{\mathbf{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right) &\leq \operatorname{Prob}\left(\left\{\|h\|_\infty > \frac{u}{a(1+O(aS))}\right\}\right) \\ &\leq 2\operatorname{Prob}\left(\left\{\|h\|_{\mathbf{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right), \end{aligned} \tag{47}$$

and the factor 2 is negligible on the *logarithmic* scale.

To evaluate

$$\operatorname{Prob}\left(\left\{\|h\|_{\mathbf{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right) \tag{48}$$

we note that (34) together with (44) imply that

$$\frac{u}{a(1+O(aS))} \to \infty, \tag{49}$$

so that we may apply Theorem 3.1 to obtain

$$\begin{aligned}&\operatorname{Prob}\left(\left\{\|h\|_{\mathbf{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right)\\ &= O\left(\exp\left[\frac{\alpha_h u}{a(1+O(aS))} - \frac{u^2}{2a^2\sigma_h^2(1+O(aS))^2}\right]\right)\end{aligned}$$

To get a lower bound for (48), we proceed as in section 3 and choose $x_0 \in S$ where $\sigma_h^2 = \sup_x r_h(x,x)$ is attained. Clearly, we shall get a lower bound in (48) by evaluating $\operatorname{Prob}\left(\left\{h(x_0) > \frac{u}{a(1+O(aS))}\right\}\right)$, and the latter is equal to $\Psi(u/(a\sigma_h^2(1+O(aS))))$.

Next, we remark that $u/(a(1+O(aS))) \sim u/a$ provided S is chosen so that $aS = o(1)$. Comparing the estimates from above and from below, we find that

$$\log \operatorname{Prob}\left(\left\{\|h\|_{\mathbf{T}^2} > \frac{u}{a(1+O(aS))}\right\}\right) = \frac{-u^2}{2a^2\sigma_h^2}.$$

This concludes the proof of Theorem 5.2 for $\Sigma = \mathbf{T}^2$, provided (41) holds (that ensures that the last term will give the dominant contribution to $\operatorname{Prob}(A_{u,a})$); it remains to show that we can choose S that will satisfy all the constraints we encountered; accordingly, we then collect all the inequalities that relate the various parameters in the course of the proof, and make sure that a proper choice for S is possible.

For the applications of Theorem 3.1, we need both $S \to \infty$ and (49); for the latter it is sufficient to require that $aS = o(1)$ or, equivalently, $S = o(1/a)$ (recall that we assume (34)). To make sure that (41) holds, we need $u/a = o(S)$. All in all, we need $u/a = o(S)$ and $S = o(1/a)$ while $S \to \infty$; the assumption $u \to 0$ of the present theorem leaves a handy margin for a possible choice of S, since it implies that u/a is much smaller than $1/a$.

Step 2b.

We next consider the case $\Sigma = \mathcal{S}^2$ or $\sigma = S_\gamma, \gamma \geq 2$. We want to estimate the probability of the event $\{||R_1 - R_0||_\infty > u\} \cap B_S^c$. Recall from (43) that

$$R_1 - R_0 = R_0(e^{-af} - 1) - ae^{-af}h.$$

By the definition of B_S, on B_S^c, we have for $x \in \Sigma$, $|f(x)| = O(S)$ and

$$|h(x)| = |\Delta_0 f(x)| = O(S).$$

Again, we choose S so that $aS = o(1)$, and it follows easily from the Taylor expansion of e^{-af} and the definition of w that

$$R_1 - R_0 = -aw - O(aS)(af + ah) = -aw + O(a^2S^2). \tag{50}$$

On S^2, the isotropic random field w has constant variance σ_w^2 that will be computed later; on $S_\gamma, \gamma \geq 2$ the variance $r_w(x,x)$ is no longer constant, and we denote by σ_w^2 its supremum $\sup_{x\in S_\gamma} r_w(x,x)$.

Therefore (cf. (45))

$$\begin{aligned}\operatorname{Prob}(A_{u,a} \cap B_S^c) &= \operatorname{Prob}\left(\left\{\|w + O(aS^2)\|_\infty > \frac{u}{a}\right\} \cap B_S^c\right)\\ &= \operatorname{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right) + O(\operatorname{Prob}(B_S)).\end{aligned}$$

Assuming that (41) holds and taking (40) into account, we obtain

$$\begin{aligned}\mathrm{Prob}(A_{u,a}) &= \mathrm{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right)\\ &+\ O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right).\end{aligned} \tag{51}$$

We choose S so that $\frac{u}{a} = o(S)$ but $S = o\left(\frac{\sqrt{u}}{a}\right)$, so that this choice is possible since $\sqrt{u}$ is much larger than u, as u is small. We then have

$$aS^2 = o\left(\frac{u}{a}\right),$$

so that

$$\mathrm{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right) = \mathrm{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a}(1+o(1))\right\}\right).$$

As in section 3, we shall estimate the quantity $\mathrm{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a}(1+o(1))\right\}\right)$ from above and below by separate arguments. We let

$$\tau = \tau(u,a,S) := u/a + O(aS^2) = (u/a)(1+o(1)).$$

By Borel-TIS Theorem 3.1, there exists α_w such that

$$\mathrm{Prob}\left(\{||w||_\infty > \tau\}\right) \le \exp\left(\alpha_w \tau - \frac{\tau^2}{2\sigma_w^2}\right). \tag{52}$$

This concludes the proof of the upper bound in (35) in this case.

To get a lower bound in (35), consider the point $x_0 \in \Sigma$ where $r_w(x,x)$ attains its maximum, $r_w(x_0,x_0) = \sigma_w^2$. Consider the event $\{|w(x_0)| > \tau\}$. We find that trivially

$$\mathrm{Prob}\left(\{||w||_\infty > \tau\}\right) \ge \mathrm{Prob}(\{|w(x_0)| > \tau\}) \ge \left(\frac{C_1}{\tau} - \frac{C_2}{\tau^3}\right)\exp\left(-\frac{\tau^2}{2\sigma_w^2}\right). \tag{53}$$

We next pass to the limit $u \to 0, u/a \to \infty$; then $\tau \cdot a/u \to 1$. Taking logarithm in (52) and (53) and comparing the upper and lower bound, we establish (35) for surfaces of genus ≥ 2. This concludes the proof of Theorem 5.2.

□

6. Dimension $n > 2$

Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$. Let $R_0 \in C^0(M)$ be the scalar curvature of g_0; we assume that R_0 has constant sign.

Let $g_1 = e^{af} g_0$ with f as in (2) be a conformal change of metric. The key difference between dimension 2 and dimension $n > 2$ in our calculations is the presence of the (non-Gaussian) gradient term $a^2(n-1)(n-2)|\nabla_0 f|^2/4$ in the equation (8). We shall assume that $c_j = O(\lambda_j^{-s}), s > n/2+1$. Then $R_1 \in C^0(M)$ a.s. by Proposition 2.5.

Below, we shall consider the random field $v(x) = (\Delta_0 f)(x)/R_0(x)$. As usual, we let

$$\sigma_v^2 = \sup_{x\in M} r_v(x,x). \tag{54}$$

Let $P_2(a)$ be the probability of the scalar curvature sign change after the conformal metric transformation $g_1 = e^{af} g_0$, i.e.

$$P_2(a) := \text{Prob}\{\exists x \in M : \operatorname{sgn} R_1(x) \neq \operatorname{sgn} R_0(x)\}.$$

6.1. **Negative R_0.** We shall first consider the case of $\forall x \in M.\, R_0(x) < 0$.

Proposition 6.1. *Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$, such that the scalar curvature $R_0 \in C^0(M)$ and $\forall x \in M.\, R_0(x) < 0$. Assume that $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, so that $h, R_1 \in C^0(M)$. Then there exists $\alpha > 0$ so that*

$$P_2(a) = O\left(\exp\left(\frac{\alpha}{a(n-1)} - \frac{1}{2a^2(n-1)^2\sigma_v^2}\right)\right).$$

Proof of Proposition 6.1.

Recall that the curvature transformation corresponding to the conformal change $g_1 = e^{af} g_0$ is given by (8), and observe that $R_1 e^{af}$ is not Gaussian because of the presence of the non-Gaussian term $|\nabla_0 f|^2$. Then, in order to prove the assertion of the present proposition, we will get rid of the term $|\nabla_0 f|^2$ in (8) by taking advantage of its positivity, so that

$$\{R_0 - a(n-1)\Delta_0 f > 0\} \supseteq \{R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4 > 0\}.$$

Therefore

$$P_2(a) \leq \text{Prob}\{\exists x \in M.\, R_0(x) - a(n-1)(\Delta_0 f)(x) > 0\}.$$

Recall that $h = \Delta_0 f$. We remark that $\operatorname{sgn}(R_0(x) - a(n-1)h(x)) = -\operatorname{sgn}(1 - (n-1)h(x)/R_0(x))$. Accordingly,

$$P_2(a) \leq \text{Prob}\{\exists x \in M.\, 1 - a(n-1)h/R_0 < 0\} = \text{Prob}\{||h/R_0||_M > 1/(a(n-1))\}.$$

It then remains to apply Theorem 3.1 for $u = 1/(a(n-1))$.

□

6.2. **Positive R_0.** We next consider the more involved case $R_0 > 0$. The regularity assumptions are the same as in section 6.1.

In this section, we shall consider the random field $v = h/R_0$ (considered earlier in section 6.1). We shall also consider the quantity

$$\sigma_2 = \sup_{x \in M} \frac{\mathbb{E}[|\nabla_0 f(x)|^2]}{R_0(x)}. \tag{55}$$

Proposition 6.2. *Let (M, g_0) be a compact orientable n-dimensional Riemannian manifold, $n > 2$, such that the scalar curvature $R_0 \in C^0(M)$ and $\forall x \in M.\, R_0(x) > 0$. Assume that $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, so that $h, R_1 \in C^0(M)$. Then there exists $\beta > 0$ so that*

$$P_2(a) = O\left(\exp\left(\frac{\beta}{a} - \frac{B}{a^2}\right)\right),$$

where

$$B = \frac{2 + \kappa - \sqrt{\kappa^2 + 4\kappa}}{\sigma_2 n(n-1)(n-2)}.$$

and

$$\kappa = \frac{4\sigma_v^2(n-1)}{\sigma_2 n(n-2)}.$$

Proof of Proposition 6.2.

Note that $\operatorname{sgn} R_1$ is equal to

$$\operatorname{sgn}\left(1 - \frac{a(n-1)h}{R_0} - \frac{a^2(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right);$$

here we have used the assumption that $\forall x \in M.\, R_0(x) > 0$. Recall that $P_2(a)$ denotes the probability that $\exists x \in M : R_1(x) < 0$. We define a random field u to be

$$u := \frac{(n-1)h}{R_0} + \frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}.$$

Then $R_1 < 0$ is equivalent to

$$\|u\|_M > \frac{1}{a}.$$

For every $0 \le \delta \le 1$, we have

$$\left\{\|u\|_M \ge \frac{1}{a}\right\} \subseteq \left\{\left\|\frac{(n-1)h}{R_0}\right\|_M \ge \frac{\delta}{a}\right\} \cup \left\{\left\|\frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right\|_M \ge \frac{1-\delta}{a}\right\},$$

so that

$$\text{Prob}\left\{\|u\|_M \ge \frac{1}{a}\right\} \le \text{Prob}\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M \ge \frac{\delta}{a}\right\} + \text{Prob}\left\{\left\|\frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right\|_M \ge \frac{1-\delta}{a}\right\}. \tag{56}$$

The probability $\text{Prob}\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M \ge \frac{\delta}{a}\right\}$ can be estimated in a straightforward way using Theorem 3.1. Indeed, define the random field $v := h/R_0$ (as in section 6.1); as before, let σ_v^2 be defined by (54). Then $\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M \ge \frac{\delta}{a}\right\}$ is equivalent to

$$\|v\|_M > \frac{\delta}{a(n-1)},$$

and the latter can be bounded by Theorem 3.1 (letting $u = \delta/(a(n-1))$) as

$$\text{Prob}\left\{\left\|\frac{(n-1)h}{R_0}\right\|_M \ge \frac{\delta}{a}\right\} \le \exp\left(\frac{\beta_1 \delta}{a} - \frac{\delta^2}{2(a(n-1))^2\sigma_v^2}\right), \tag{57}$$

for some constant $\beta_1 > 0$.

To bound

$$\text{Prob}\left\{\left\|\frac{a(n-1)(n-2)|\nabla_0 f|^2}{4R_0}\right\|_M \ge \frac{1-\delta}{a}\right\} = \text{Prob}\left\{\left\|\frac{|\nabla_0 f|^2}{R_0}\right\|_M \ge \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\}$$

we need to work harder, as the random field $|\nabla_0 f(x)|^2/R_0(x)$ is not Gaussian. The key observation is that we may represent this field as *locally* Gaussian subordinated. Namely, let

$$\{U_i : i = 1, \dots, m\}$$

be a finite covering of M so that there exists a geodesic frame $\{E_1^i, \dots, E_n^i\}$ defined on U_i. On U_i we have

$$\frac{|\nabla_0 f(x)|^2}{R_0(x)} = \sum_{k=1}^{n} \frac{(E_k^i f(x))^2}{R_0(x)},$$

and we observe that $G_{i,k}(x) := (E_k^i f(x))/\sqrt{R_0(x)}$ are *centered Gaussian* random fields defined on U_i. For each i, k and $x \in M$ we have

$$\mathbb{E}[G_{i,k}(x)^2] \le \mathbb{E}\left[\frac{|\nabla f(x)|^2}{R_0(x)}\right] \le \sigma_2 \tag{58}$$

by the definition (55).

We then have

$$\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_M = \max_i \left\|\sum_{k=1}^{n} G_{i,k}(x)^2\right\|_{U_i},$$

so that

$$\begin{aligned} &\text{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_M \ge \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\} \\ &\le \sum_{i=1}^{m} \text{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_i} \ge \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\} \\ &\le m\text{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_{i_0}} \ge \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\} \end{aligned} \tag{59}$$

where $i_0 = i_0(a)$ maximizes the probability

$$\text{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_i} \ge \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\}$$

for $1 \le i \le m$.

Therefore we need to bound

$$\begin{aligned} &\text{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_{U_{i_0}} \ge \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\} \\ &= \text{Prob}\left\{\left\|\sum_{k=1}^{n} \frac{(E_k^{i_0} f(x))^2}{R_0(x)}\right\|_{U_{i_0}} \ge \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\} \\ &\le \sum_{k=1}^{n} \text{Prob}\left\{\left\|\frac{|E_k^{i_0} f(x)|}{\sqrt{R_0(x)}}\right\|_{U_{i_0}} \ge \sqrt{\frac{4(1-\delta)}{a^2 n(n-1)(n-2)}}\right\}. \end{aligned} \tag{60}$$

We may bound each of the summands using the Borel-TIS inequality as

$$\begin{aligned} &\text{Prob}\left\{\left\|\frac{|(E_k^{i_0} f(x))|}{\sqrt{R_0(x)}}\right\|_{U_{i_0}} \ge \sqrt{\frac{4(1-\delta)}{a^2 n(n-1)(n-2)}}\right\} \\ &\le \exp\left(\frac{\beta_2}{a} - \frac{2(1-\delta)}{a^2\sigma_2 n(n-1)(n-2)}\right), \end{aligned}$$

where we exploited (58); the constant β_2 absorbs the 2 factor coming from the possibility that we might have either a positive or negative sign. Plugging the last estimate into (60) and the resulting bound into (59), we finally obtain, possibly choosing a larger constant β_2 to absorb the constants in front of the exponent,

$$\begin{aligned}&\text{Prob}\left\{\left\|\frac{|\nabla_0 f(x)|^2}{R_0(x)}\right\|_M \geq \frac{4(1-\delta)}{a^2(n-1)(n-2)}\right\}\\ &\leq \exp\left(\frac{\beta_2}{a}-\frac{2(1-\delta)}{a^2\sigma_2 n(n-1)(n-2)}\right).\end{aligned} \tag{61}$$

We next choose δ in an optimal way, so that the negative exponents in (61) and (57) match, i.e. so that

$$\frac{\delta^2}{2(n-1)^2\sigma_v^2}=\frac{2(1-\delta)}{\sigma_2 n(n-1)(n-2)}. \tag{62}$$

or, letting $\kappa=\frac{4\sigma_v^2(n-1)}{\sigma_2 n(n-2)}$,

$$\delta^2+\kappa\delta-\kappa=0.$$

It is easy to check that the root $\delta_0=(\sqrt{\kappa^2+4\kappa}-\kappa)/2$ satisfies the required inequality $0<\delta<1$ and thus gives an admissible solution to (62). Substituting δ_0, we find that the exponents in (62) are both equal to

$$B=\frac{2+\kappa-\sqrt{\kappa^2+4\kappa}}{\sigma_2 n(n-1)(n-2)}.$$

Substituting into (61) and (57) finishes the proof of Proposition 6.2.

□

7. Q-curvature

The Q-curvature was first studied by Branson and later by Gover, Orsted, Fefferman, Graham, Zworski, Chang, Yang, Djadli, Malchiodi and others. We refer to [BG] for a detailed survey.

7.1. **Conformally covariant operators.** Here we summarize some useful results in [BG]. Let M be a manifold of dimension $n\geq 3$. Let m be even, and $m\notin\{n+2,n+4,\dots\}\Leftrightarrow m-n\notin 2\mathbf{Z}^+$. Then there exists on M an elliptic operator P_m (GJMS operators of Graham-Jenne-Mason-Sparling, cf. [GJMS]).

We shall restrict ourselves to even n, and to $m=n$. We shall denote the corresponding operator P_n simply by P. It satisfies the following properties: $P=\Delta^{n/2}+$ *lower order terms*. P is formally self-adjoint (Graham-Zworski [GZ], Fefferman-Graham [FG]). Under a conformal change of metric $\tilde g=e^{2\omega}g$, the operator P changes as follows: $\widetilde{P}=e^{-n\omega}P$. P has a polynomial expression in (Levi-Civita connection) ∇ and (scalar curvature) R, with coefficients that are rational in dimension n.

The operator $P_4=\Delta_g^2+\delta[(2/3)R_g g-2\mathrm{Ric}_g]d$ is called the *Paneitz operator*.

7.2. **Q-curvature and its key properties.** We shall henceforth only consider manifolds of even dimension n. The Q-curvature in dimension 4 was defined by Paneitz as follows:

$$Q_g=-\frac{1}{12}\left(\Delta_g R_g-R_g^2+3|\mathrm{Ric}_g|^2\right). \tag{63}$$

In higher dimensions, Q-curvature is a local scalar invariant associated to the operator P_n. It was introduced by T. Branson in [Bran]; alternative constructions were provided in [FG, FH] using the *ambient metric* construction.

Q-curvature is equal to $1/(2(n-1))\Delta^{n/2}R$ modulo nonlinear terms in curvature. Under a conformal change of variables $\tilde{g} = e^{2\omega}g$ on M^n, the Q-curvature transforms as follows [BG, (4)]:

$$P\omega + Q = \tilde{Q}e^{n\omega}. \tag{64}$$

Integral of the Q-curvature is conformally invariant.

A natural problem is the existence of metrics with constant Q-curvature in a given conformal class. In the following proposition, we summarize results due to Chang and Yang, and Djadli and Malchiodi in dimension 4, and to Ndiaye in arbitrary even dimension $n > 4$ [CY, DM, N].

Proposition 7.1. *Let (M, g) be a compact Riemannian manifold of even dimension $n \geq 4$, and assume that M satisfies the following "generic" assumptions:*

i) *In dimension $n = 4$, the assumptions are ([DM]): $\ker P_n = \{const\}$, and $\int_M QdV \neq 8\pi^2 k, k = 1, 2, \ldots$*

ii) *In even dimension $n > 4$, the assumptions are ([N]): $\ker P_n = \{const\}$, and $\int_M QdV \neq (n-1)!\omega_n k, k = 1, 2, \ldots,$ where $(n-1)!\omega_n = \int_{S^n} QdV$, the integral of Q-curvature for the round $\mathcal{S}^n$.*

Then there exists a metric g_Q on M in the conformal class of g with constant Q-curvature. If $n = 4, \int_M QdV < 8\pi^2$, $P_4 \geq 0$ and $\ker P_4 = \{const\}$, then g_Q is unique, [CY, Thm 2.2].

If g has positive scalar curvature and $M \neq \mathcal{S}^4$, then the assumption $\int_M QdV < 8\pi^2$ is satisfied; if in addition $\int_M Q \geq 0$, then the assumptions $P_4 \geq 0$ and $\ker P_4 = \{const\}$ are also satisfied.

7.3. **Generalizing the results for scalar curvature.** We explain the strategy to generalize our results for scalar curvature to Q-curvature. We consider a manifold M with a "reference" metric g_0 such that Q-curvature has constant sign and a conformal perturbation $g_1 = e^{2af}g_0$ where a is a positive number; we expand f in a series of eigenfunctions of P. Next, we use formula (64) to study the induced curvature Q_1. Finally, we use methods of Adler-Taylor to prove sharper estimates for the probability for homogeneous manifolds with constant Q-curvature.

We remark that in every conformal class where the generic conditions of [DM, N] hold, there exist metrics with Q-curvature of constant sign.

7.4. **Q-curvature in a conformal class.** Let M be a manifold of even dimension n, and let g_0 be a metric with Q-curvature Q_0.

In the Fourier expansions considered below, we shall restrict our summation to *nonzero* eigenvalues of P_n. We remark that the assumptions of Proposition 7.1, then $\ker P_n = \{const\}$.

Let $P = P_n$ have k negative eigenvalues (counted with multiplicity); denote the corresponding spectrum by $P\psi_j = -\mu_j\psi_j$, for $1 \leq j \leq k$, where $0 > -\mu_1 \geq -\mu_2 \geq \ldots \geq -\mu_k$. The other nonzero eigenvalues are positive, and the corresponding spectrum is denoted by $P\phi_j = \lambda_j\phi_j$, for $j \geq 1$, where $0 < \lambda_1 \leq \lambda_2 \leq \ldots$.

Consider the change of metric $g_1 = e^{2af}g_0$, where we let

$$f = \sum_{i=1}^{k} b_i\psi_i + \sum_{j=1}^{\infty} a_j\phi_j, \tag{65}$$

and where $b_i \sim \mathcal{N}(0, t_i^2)$ and $a_j \sim \mathcal{N}(0, c_j^2)$.

We define $h := -Pf$, and substituting into (64), we find that

$$Q_1 e^{naf} = Q_0 - ah = Q_0 + a\left(\sum_{j=1}^{\infty} \tilde{a}_j \phi_j - \sum_{i=1}^{k} \tilde{b}_i \psi_i\right), \tag{66}$$

where $\tilde{a}_j \sim \mathcal{N}(0, \lambda_j^2 c_j^2)$ and $\tilde{b}_i \sim \mathcal{N}(0, t_i^2 \mu_i^2)$.

Remark 7.2. *It follows that $Q_1 e^{naf}(x)$ is Gaussian with expectation $Q_0(x)$ and covariance function*

$$a^2 \cdot r_h(x,y) = a^2 \left(\sum_{i=1}^{k} t_i^2 \mu_i^2 \psi_i(x)\psi_i(y) + \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 \phi_j(x)\phi_j(y)\right). \tag{67}$$

7.5. **Regularity.** It is easy to see that the regularity of the random field in (65) is determined by the principal symbol $\Delta^{n/2}$ of the GJMS operator $P = P_n$. The following Proposition is then a straightforward extension of Proposition 2.1:

Proposition 7.3. *Let f be defined as in* (65). *If $c_j = O(\lambda_j^{-t})$ and $t > 1 + \frac{k}{n}$, then $f \in C^k$. Similarly, if $c_j = O(\lambda_j^{-t})$ and $t > 2 + \frac{k}{n}$ then $Pf \in C^k$.*

7.6. **Using Borel-TIS to estimate the probability that Q-curvature changes sign.** Consider a metric g_0 where $Q_0(x)$ has constant sign. We remark that such metric always exists in the conformal class of g_0 if Proposition 7.1 holds.

Let f be as in equation (65) and such that Pf is a.s. C^0. We remark that it follows from Proposition 7.3 that this happens if $c_j = O(\lambda_j^{-t})$ where $t > 2$.

Let $g_1 = e^{2af} g_0$. Denote the Q-curvature of g_1 by Q_1; then it follows from (64) that

$$\operatorname{sgn}(Q_1) = \operatorname{sgn}(Q_0)\operatorname{sgn}(1 - ah/Q_0) \tag{68}$$

It follows that Q_1 changes sign iff $\sup_{x\in M} h(x)/Q_0(x) > 1/a$.

We denote by $v(x)$ the random field $h(x)/Q_0$. It follows from (67) that the covariance function of $v(x)$ is equal to

$$r_v(x,y) = \frac{1}{Q_0(x)Q_0(y)} \left(\sum_{i=1}^{k} t_i^2 \mu_i^2 \psi_i(x)\psi_i(y) + \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 \phi_j(x)\phi_j(y)\right). \tag{69}$$

We let

$$\sigma_v^2 := \sup_{x\in M} r_v(x,x). \tag{70}$$

As for the scalar curvature, we make the following

Definition 7.4. *Denote by $P_2(a)$ the probability that the Q-curvature Q_1 of the metric $g_1 = g_1(a)$ changes sign.*

Theorem 7.5. *Assume that $Q_0 \in C^0(M)$ and that $c_j = O(\lambda_j^{-t}), t > 2$. Then there exist constants $C_1 > 0$ and C_2 such that the probability $P_2(a)$ satisfies*

$$(C_1 a)e^{-1/(2a^2\sigma_v^2)} \le P_2(a) \le e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as $a \to 0$. In particular

$$\lim_{a\to 0} a^2 \ln P_2(a) = \frac{-1}{2\sigma_v^2}.$$

Proof of Theorem 7.5. It follows from the assumptions of the theorem and from Proposition 7.3 that $v \in C^0(M)$ a.s., and hence the Borell-TIS theorem applies. The rest of the proof follows the proof of Theorem 3.3.

□

7.7. **L^∞ bounds for the Q-curvature.** Here we extend the results in section 5 to Q-curvature. We have not pursued similar questions for the scalar curvature in dimension $n \geq 3$ due to the presence of the gradient term in the transformation formula (8). For the Q-curvature, there is no gradient term in the corresponding transformation formula (64), which allows us to establish the following

Theorem 7.6. *Let (M, g_0) be an n-dimensional compact orientable Riemannian manifold, with n even. Assume that $Q_0 \in C^0(M)$, and that $c_j = O(\lambda_j^{-t})$, $t > 2$, so that by Proposition 7.3 the random fields f and h are a.s. C^2. Let $w := h - nQ_0 f$, denote by $r_w(x, y)$ its covariance function and set*

$$\sigma_w^2 := \sup_{x \in M} r_w(x, x).$$

Let $a \to 0$ and $u \to 0$ so that

$$\frac{u}{a} \to \infty.$$

Then

$$\log \operatorname{Prob}(\|Q_1 - Q_0\|_\infty > u) \sim -\frac{u^2}{2a^2\sigma_w^2}.$$

Proof. The proof of this theorem is very similar to the one presented in step (2c) of Theorem 5.2. We will have to deal with the fact that neither f, h or w have constant variance. We start by defining the "bad" event B_S, for $S > 0$,

$$B_S = \{||f||_\infty > S\} \cup \{||h||_\infty > S\}.$$

By Theorem 3.1, there exist two constants α_f and α_h such that

$$\operatorname{Prob}(B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right).$$

for $\sigma_f^2 := \sup_{x \in M} r_f(x, x)$ and $\sigma_h^2 := \sup_{x \in M} r_h(x, x)$, where r_f and r_h are the covariance functions of f and h respectively.
As before, we denote $A_{u,a}$ the event $\{||Q_1 - Q_0||_\infty > u\}$ and observe that $\operatorname{Prob}(A_{u,a}) = \operatorname{Prob}(A_{u,a} \cap B_S) + \operatorname{Prob}(A_{u,a} \cap B_S^c)$. We estimate $\operatorname{Prob}(A_{u,a} \cap B_S)$ trivially : $\operatorname{Prob}(A_{u,a} \cap B_S) \leq \operatorname{Prob}(B_S)$. This implies that

$$\operatorname{Prob}(A_{u,a} \cap B_S) = O\left(\exp\left(\alpha_f S - \frac{S^2}{2\sigma_f^2}\right) + \exp\left(\alpha_h S - \frac{S^2}{2\sigma_h^2}\right)\right).$$

In order to estimate $\operatorname{Prob}(A_{u,a} \cap B_S^c)$, recall that in the case of Q-curvature, it follows from (64) that

$$Q_1 - Q_0 = Q_0(e^{-naf} - 1) + ahe^{-naf}.$$

By the definition of B_S, on B_S^c, we have for $x \in M$ that

$$|f(x)| = O(S) \quad \text{and} \quad |h(x)| = O(S).$$

We choose S so that $aS = o(1)$. It follows easily from the Taylor expansion of e^{-af} and the definition of w that

$$Q_1 - Q_0 = ah - anQ_0 f + O(a^2S^2) = aw + O(a^2S^2).$$

Therefore,

$$\begin{aligned}\mathrm{Prob}(A_{u,a} \cap B_S^c) &= \mathrm{Prob}\left(\left\{\|w + O(aS^2)\|_\infty > \frac{u}{a}\right\} \cap B_S^c\right)\\ &= \mathrm{Prob}\left(\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\}\right) + O(\mathrm{Prob}(B_S)).\end{aligned}$$

The notation was conveniently chosen so that the rest of the proof be identical to that of Theorem 5.2 step (2c).

□

8. Conclusion

In the present paper, we considered a random conformal perturbation g_1 of the reference metric g_0 (which we assumed to have constant scalar curvature R_0), and studied the following questions about the scalar curvature R_1 of the new metric:

i) Assuming $R_0 \neq 0$, estimate the probability that R_1 changes sign;
ii) Estimate the probability that $||R_1 - R_0||_\infty > u$, where $u > 0$ is a parameter.

We also studied analogous questions for Branson's Q-curvature.

The measures on metrics in a conformal class that we define "localize" to a reference metric g_0 in the limit $a \to 0$, where g_0 is the Yamabe metric when we study the scalar curvature; or the metric with $Q_0 \equiv \mathit{const}$ when we study the Q-curvature.

In the proofs, we used Theorem 4.6 due to Adler-Taylor, or (Borel-TIS) Theorem 3.1, which requires lower regularity of random fields, but gives less precise estimates.

8.1. **Further questions.** There are numerous questions that were not addressed in the present paper. We concentrated on he study of *local* geometry of spaces of positively- or negatively-curved metrics (see Remark 2.4), but it seems extremely interesting to study *global* geometry of these spaces, [GL, Kat, Lo, Ros06, Sch87, SY79-1, SY82, SY87].

Another interesting question that seems tractable concerns the study of the *nodal set* of R_1 i.e. its zero set. That set, like the sign of R_1, only depends on the quantity $R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4$ (or $R_0 - a\Delta_0 f$ in dimension two). It also seems interesting to study other characteristics of the curvature (whether it changes sign or not), such as its L^p norms, the structure of its nodal domains (if it changes sign), and of its sub- and super-level sets.

Also, it seems quite interesting to study related questions for Ricci and sectional curvatures in dimension $n \geq 3$.

Another important question concerns an appropriate definition of measures on the space of Riemannian metrics not restricted to a single conformal class.

A very important question concerns the study of metrics of lower regularity than in the present paper, appearing e.g. in 2-dimensional quantum gravity, cf. [DS].

In addition, it seems very interesting to study various questions about random metrics that are influenced by curvature, such as various geometric invariants (girth, diameter, isoperimetric constants etc); spectral invariants (small eigenvalues of Δ, determinants of Laplacians, estimates for the heat kernel, statistical properties of eigenvalues and of the spectral function, etc); as well as various questions related to the geodesic flow or the frame flow on M, such as existence of conjugate points, ergodicity, Lyapunov exponents and entropy, etc.

We plan to address these and other questions in subsequent papers.

Appendix A. Metrics with positive and negative scalar curvature

In this section we review some results about the spaces of metrics of positive and negative scalar curvature. In dimension two, S^2 admits the metric of positive curvature, and surfaces of genus ≥ 2 admit metrics of negative curvature. For connected manifolds M of dimension $n \geq 3$, Kazdan and Warner proved the following "trichotomy" theorem:

i) If M admits a metric of nonnegative and not identically 0 scalar curvature, then any $f \in C^\infty(M)$ can be realized as a scalar curvature of some Riemannian metric.
ii) If M is not in (i) and admits a metric of vanishing scalar curvature, then $f \in C^\infty(M)$ can be realized as a scalar curvature provided $f(x) < 0$ for some $x \in M$, or else $f \equiv 0$.
iii) If M is not in (i) or (ii), then $f \in C^\infty(M)$ can be realized as a scalar curvature provided $f(x) < 0$ for some $x \in M$.

A.1. **Negative scalar curvature.** Denote by $S^-(M)$ the space of metrics of negative scalar curvature on a manifold M of dimension $n \geq 3$; it follows from results of Aubin and Kazdan-Warner that $S^-(M)$ is always nonempty. A fundamental theorem about the structure of $S^-(M)$ was proved by J. Lohkamp [Lo], who showed that $S^-(M)$ is connected and aspherical (and hence is contractible). He also showed that the space $S_{-1}(M)$ of metrics of constant curvature -1 is contractible. It is shown in [Kat] that on a Haken manifold, the moduli space $S_{-1}(M)/Diff_0(M)$ (where $Diff_0(M)$ denotes the the group of diffeomorphisms isotopic to the identity) is also contractible, similarly to Teichmuller spaces for surfaces of genus ≥ 2 in dimension 2.

In a different paper, Lockhamp showed that $S^-(M)$ and $S_{-1}(M)$ are never convex.

A.2. **Positive scalar curvature.** Questions about existence and spaces of metrics of positive scalar curvature are more complicated than similar questions for negative scalar curvature. Here we recall some of the less technical results in recent Rosenberg's survey [Ros06]. We make no attempt to give a complete survey, we just want to list some examples of manifolds where the results of our paper hold.

There are several techniques for proving results about non-existence of metrics of positive scalar curvature on a given manifold. We assume that M is compact, closed, oriented manifold.

i) For spin manifolds with positive scalar curvature, it follows from the work of Lichnerowicz that all harmonic spinors (lying in the kernel of the Dirac operator) have to vanish; it follows from the work of Lichnerowicz and Hitchin that any manifold with nonvanishing Hirzebruch genus $\hat{A}(M)$ has no metrics of positive scalar curvature. We refer to [Ros86] and [Ros06] for further non-existence results that use index theory of Dirac operator, and for relations to Novikov conjectures.
ii) It follows from the work of Schoen and Yau on minimal surfaces [SY79-1, SY79-2, SY82] that if N is a stable $(n-1)$-dimensional submanifold of an

n-dimensional manifold M with positive scalar curvature, and if N dual to a nonzero element in $H^1(M, \mathbf{Z})$, then N also admits a metric of positive scalar curvature. It was shown in [SY79-2] that if on a 3-manifold $\pi_1(M)$ contains a product of two cyclic groups, or a subgroup isomorphic to the fundamental group of a compact Riemann surface of genus > 1, then M cannot have a metric of positive scalar curvature. Moreover, it was shown in [SY87] that a closed aspherical 4-manifold cannot admit a metric of positive scalar curvature.

iii) Furhter negative results for 4-manifolds can be obtained using Seiberg-Witten theory. It was shown by Witten and Morgan that on a 4-manifold with $b_2^+(M) > 1$, if the Seiberg-Witten invariant $SW(\xi) \neq 0$ for some $spin^c$ structure ξ, then M does not admit a metric of positive scalar curvature. Taubes showed that existence of a symplectic structure on a 4-manifold with $b_2^+(M) > 1$ implies the previous condition. We refer to [Ros06] for a summary of results in case $b_2^+(M) = 1$.

In the positive direction, it was shown by Gromov-Lawson and Schoen-Yau [GL, SY79-1] that if M_0 is a manifold (not necessarily connected) of positive scalar curvature, then any manifold M_1 obtained from M_0 by a surgery in codimension ≥ 3 also admits a metric of positive scalar curvature. In dimension $n \geq 5$, the condition $w_2(M) \neq 0$ (where $w_2(M)$ is the second Stiefel-Whitney class of M) implies the existence of metrics with positive scalar curvature.

A.3. **Moduli spaces of metrics of positive scalar curvature.** Denote by $S^+(M)$ the space of metrics of negative scalar curvature on a manifold M of dimension $n \geq 3$ (the space $S^+(S^2)$ is contractible). In general, $S^+(M)$ is not connected. For example, Hitchin [Hit] showed that on a n-dimensional spin manifold M admitting a metric of positive scalar curvature, $\pi_0(S^+(M)) \neq 0$ if $n \equiv 0$ or $1 \mod 8$, and $\pi_1(S^+(M)) \neq 0$ if $n \equiv 0$ or $-1 \mod 8$. For more general results, we refer to the results of Stolz [Ros06, Thm. 2.3]. Gromov and Lawson proved that $S^+(S^7)$ has infinitely many components. The same result holds for $M = S^{4k-1}, k > 2$, cf. [Ros06]. we refer to [Ros06, Thm. 2.7, 2.8] for further results. In dimension 4, Ruberman showed that there exists a simply-connected M^4 with infinitely many metrics of positive scalar curvature that are *concordant* (i.e. restrictions to $s = 0$ and $s = 1$ of a metric of positive scalar curvature on $M \times [0, 1]$), but not isotopic.

Appendix B. Yamabe Problem

Yamabe problem concerns finding metrics with constant scalar curvature in a conformal class of metrics on a manifold of dimension $n > 3$. The problem was formulated by H. Yamabe [Yam] (who also claimed to solve it, but the proof contained gaps), and solved by Trudinger, Aubin and Schoen [Tr, Au76, Sch84].

The corresponding smooth metric g_γ (called *Yamabe metric*) exists in every conformal class, and minimizes the total scalar curvature

$$(\mathrm{vol(M,g)})^{-(n-2)/n} \int_M R(g) dV(g),$$

when g is restricted to a conformal class of metrics. The *sign* of $R(g_\gamma)$ is uniquely determined by the conformal class. The conformal class is called *positive* (resp. *negative*) if $R(g_\gamma) > 0$ (resp. < 0); *non-positive* and *non-negative* conformal classes are defined similarly.

The Yamabe metric is unique in every non-positive conformal class, [And05, §1]. The space $\mathcal{Y}^-(M)$ of all negative unit volume Yamabe metrics on M forms a smooth, infinite dimensional manifold, transverse to the space of conformal classes, in the space of all unit volume metrics on M, [Bes, Sch87]. Here the spaces of $\mathcal{Y}^-(M)$ (resp. $\mathcal{Y}^+(M), \mathcal{Y}^0(M)$) of metrics of negative (resp. positive, zero) scalar curvature are discussed in Appendix A.

The scalar curvature defines a smooth function $R : \mathcal{Y}^- \to \mathbf{R}$, whose critical points are Einstein metrics on M with negative scalar curvature. Similar results hold for non-positive conformal classes, [And05, §1].

The situation is much more complicated for positive conformal classes: there the Yamabe metrics are not unique in general, e.g. on S^n, the group of Möbius transformations acts on the space of Yamabe metrics. However, they are unique *generically* (for an open dense set of metrics in the space of positive conformal classes), cf. [And05, Thm. 1.1].

Appendix C. Validity of applying Adler-Taylor for $h = \Delta_0 f$ on $\mathcal{S}^2$

In this appendix we justify the application of two results. In section C.1 we prove that h satisfies the conditions of Theorem 4.6 due to Adler-Taylor, namely that h is attainable. In section C.2 we prove that the sufficient conditions for (32) hold (i.e. the hypotheses of [AT08], Theorem 14.3.3).

C.1. **Attainability of $h = \Delta_0 f$ on $\mathcal{S}^2$.** The goal of the present section is to prove that the random field $h = \Delta f$ on the 2-dimensional sphere, given by (28), is attainable (cf. Definition 4.5), assuming the coefficients c_m are decaying sufficiently rapidly as

$$c_m = O(m^{-s}) \text{ for } s > 7 \tag{71}$$

(see (26) and the assumptions of Theorem 4.3).

First, Lemma 4.4 and the assumptions on the decay of c_m imply that h is C^2 a.s. In fact, from the $C^{k,\beta}$ version of the Sobolev embedding theorem (cf. [Au98, Thm. 2.10, 2nd part]), it follows from the strict inequality $s > 7$ that there exists $\beta > 0$ such that

$$h \in C^{2,\beta}(S^2) \text{ a.s.}; \tag{72}$$

this will be used later. Next we check conditions (1) and (2) of Definition 4.5 of attainability.

For each $y \in \mathcal{S}^2$ let $\delta : T \to \mathcal{S}^2$ be the spherical coordinates with pole at y, where $T = [0, \pi] \times [0, 2\pi]$. Namely, we let (θ_y, ϕ_y) be the standard spherical coordinates of y and define

$$\delta_y(\theta, \phi) = (\sin(\theta - \theta_y)\cos(\phi - \phi_y), \sin(\theta - \theta_y)\sin(\phi - \phi_y), \cos(\theta - \theta_y)),$$

and $\psi_y := \delta_y^{-1}$. Let $x \in \mathcal{S}^2$ be a point and (U_x, ψ_y) be any small chart with $x \in U_x$, $\psi_y(U_x) \subseteq \mathbb{R}^2$ for some $y \in \mathcal{S}^2$. We claim that choosing y appropriately, a sufficiently small chart U satisfies condition (1) of attainability. This is, of course, sufficient to form a finite atlas, by the compactness of the sphere.

First, at any point $t \in U_x$, the random vector

$$H(t) = (h_i(t), h_{ij}(t))_{i<j} \in \mathbb{R}^5$$

is mean zero Gaussian (here the derivatives are w.r.t. the cartesian coordinates in $\mathbb{R}^2$). Therefore we have to check that its covariance matrix $C_{H(t)} \in M_5(\mathbb{R})$ is non-degenerate; by the locality it is sufficient to check that $C_{H(x)}$ is nonsingular. The matrix $C_{H(x)}$ depends, in general, on the choice of y; we are free to choose y as we wish.

It turns out that for y for which $\phi = \frac{\pi}{2}$, $C_{H(x)}$ is of a particularly simple form. For this choice of y, we compute $C_{H(x)} = \sum\limits_{m\ge 1} c_m C_m$ (with finite entries), where the single eigenspace covariance matrices C_m are given explicitly by

$$C_m = \begin{pmatrix} \frac{E_m}{2} I_2 & 0_{2\times 3} \\ 0_{3\times 2} & \Omega^m_{3\times 3} \end{pmatrix},$$

with

$$\Omega^m_{3\times 3} = \frac{E_m}{8} \begin{pmatrix} 3E_m - 2 & E_m + 2 & \\ E_m + 2 & 3E_m - 2 & \\ & & E_m - 2 \end{pmatrix},$$

and it is then easy to use (71) in order to check that the entries of $C_{H(x)}$ are finite and the matrix is nonsingular.

Remark C.1. *A priori, it seems that non-degeneracy of H in one point is sufficient, thanks to the isotropic property of h. However, one should bear in mind that introducing a chart breaks the symmetry, so that the second derivatives are no longer isotropic, being dependent on the local properties of the corresponding frame. This is unlike the first (directional) derivatives, which depend only on the direction of the frame at the given point.*

As for condition (2) of Definition 4.5, it follows easily from (72), the latter implying

$$r_{h_{ij}}(\cdot, t), r_{h_{ij}}(t, \cdot) \in C^{0,\beta}(\mathcal{S}^2)$$

for every $t \in \mathcal{S}^2$.

C.2. **Relation of the expected Euler characteristic of the excursion set and the excursion probability.** The goal of the present section is to justify the application of [AT08], Theorem 14.3.3 on $h = \Delta f$ given by (28). Recall that the covariance function of h is r_h, given by (30).

In addition to the assumptions already validated in the previous section we are required to show that

$$r_h(x, y) = 1 \quad \Leftrightarrow \quad x = y \tag{73}$$

(recall that for every $x \in \mathcal{S}^2$ we have $r(x, x) = 1$ by the assumption (25)). This condition rules out degeneracies such as periodic processes.

We claim that (73) holds if and only if there exists an *odd* m_0 so that

$$c_{m_0} > 0. \tag{74}$$

That is guaranteed by one of the assumptions in Theorem 4.3.

To see that we note (see e.g. [W]) that for every $m \ge 1$, $|P_m(t)| \le 1$ for $t \in [-1, 1]$, $P_m(1) = 1$;

$$|P_m(t)| = 1 \Leftrightarrow t = \pm 1,$$

and P_m is even or odd, for m even or odd respectively. Thus we have by (30)

$$|r_h(x, y)| \le \sum_{m=1}^{\infty} c_m = 1$$

by (25), and the equality may hold only if $\cos(d(x,y)) = \pm 1$, i.e. $x = \pm y$. In case $x = -y$ this may not hold by (74).

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