Estimates from below for the spectral function and for the remainder in Weyl’s law

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• $X^n, n \geq 2$ - compact. $\Delta$ - Laplacian. Spectrum:
$\Delta \phi_i + \lambda_i \phi_i = 0, \ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$

**Eigenvalue counting function:**
$N(\lambda) = \# \{ \sqrt{\lambda_j} \leq \lambda \}$.

**Weyl’s law:** $N(\lambda) = C_n V \lambda^n + R(\lambda), \ R(\lambda) = O(\lambda^{n-1})$.
$R(\lambda)$ - remainder.

• **Spectral function:** Let $x, y \in X$.
$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x) \phi_i(y)$.
If $x = y$, let $N_{x,y}(\lambda) := N_x(\lambda)$.

**Local Weyl’s law:**
$N_{x,y}(\lambda) = O(\lambda^{n-1}), \ x \neq y$;
$N_x(\lambda) = C_n \lambda^n + R_x(\lambda), \ R_x(\lambda) = O(\lambda^{n-1}); \ R_x(\lambda)$ - local remainder.

• We study lower bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda)$. 
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- We study **lower** bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda).$
• Notation: $f_1(\lambda) = \Omega(f_2(\lambda))$, $f_2 > 0$ iff
  \[
  \limsup_{\lambda \to \infty} \frac{|f_1(\lambda)|}{f_2(\lambda)} > 0.
  \]
  Equivalently, $f_1(\lambda) \neq o(f_2(\lambda))$.

• **Theorem 1**[JP] If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then
  \[
  N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).
  \]

• **Theorem 2**[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then
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• Other results in dimension $n > 2$ involve heat invariants.
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• Other results in dimension \( n > 2 \) involve heat invariants.
• Example: flat square 2-torus
\[ \lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbb{Z} \]
\[ \phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}, \quad x = (x_1, x_2) \]

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**Gauss circle problem:** estimate \( R(\lambda) \).

Theorem 2 \( \Rightarrow \) \[ R(\lambda) = \Omega(\sqrt{\lambda}) \]

**Hardy–Landau bound.** Theorem 2 generalizes that bound for the local remainder.

**Soundararajan (2003):**

\[ R(\lambda) = \Omega \left( \frac{\sqrt{\lambda}(\log \lambda)^{1/4}(\log \log \lambda)^{3(2^{4/3} - 1)/4}}{(\log \log \log \lambda)^{5/8}} \right) \]

• **Hardy’s conjecture:** \( R(\lambda) \ll \lambda^{1/2+\epsilon} \forall \epsilon > 0 \).

**Huxley (2003):** \( R(\lambda) \ll \lambda^{131/208}(\log \lambda)^{2.26} \).
**Example: flat square 2-torus**

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• **Negative curvature.** Suppose sectional curvature satisfies
\[-K_1^2 \leq K(\xi, \eta) \leq -K_2^2\]

**Theorem (Berard):** $R_x(\lambda) = O(\lambda^{n-1} / \log \lambda)$

**Conjecture (Randol):** On a negatively-curved surface, $R(\lambda) = O(\lambda^{1/2+\epsilon})$. Randol proved an integrated (in $\lambda$) version for $N_{x,y}(\lambda)$.

• **Theorem (Karnaukh)** On a negatively curved surface

$$R_x(\lambda) = \Omega(\sqrt{\lambda})$$

+ logarithmic improvements discussed below.

Karnaukh’s results (unpublished 1996 Princeton Ph.D. thesis under the supervision of P. Sarnak) served as a starting point and a motivation for our work.
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• **Thermodynamic formalism:** $G^t$ - geodesic flow on $SX$, $\xi \in SX$, $T_\xi(SX) = E^s_\xi \oplus E^u_\xi \oplus E^o_\xi$,
  • $\dim E^s_\xi = n - 1$: stable subspace, exponentially contracting for $G^t$;
  • $\dim E^u_\xi = n - 1$: unstable subspace, exponentially contracting for $G^{-t}$;
  • $\dim E^o_\xi = 1$: tangent subspace to $G^t$.

**Sinai-Ruelle-Bowen potential** $\mathcal{H} : SM \to \mathbb{R}$:

$$\mathcal{H}(\xi) = \left. \frac{d}{dt} \ln \det dG^t \right|_{t=0} \bigg|_{E^u_\xi}$$

• **Topological pressure** $P(f)$ of a Hölder function $f : SX \to \mathbb{R}$ satisfies (Parry, Pollicott)

$$\sum_{l(\gamma) \leq T} l(\gamma) \exp \left[ \int_\gamma f(\gamma(s), \gamma'(s)) ds \right] \sim \frac{e^{P(f)T}}{P(f)}.$$
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• $\gamma$ - geodesic of length $l(\gamma)$. $P(f)$ is defined as
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P(f) = \sup_\mu \left( h_\mu + \int fd\mu \right),
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$\mu$ is $G^t$-invariant, $h_\mu$ - (measure-theoretic) entropy.

• Ex 1: $P(0) = h$ - topological entropy of $G^t$. Theorem (Margulis): $\# \{ \gamma : l(\gamma) \leq T \} \sim e^{hT}/hT$.

Ex. 2: $P(-H) = 0$.

• Theorem 3 [JP] If $X$ is negatively-curved then for any $\delta > 0$ and $x \neq y$
\[
N_{x,y}(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right)
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Here $P(-\mathcal{H}/2)/h \geq K_2/(2K_1) > 0$. 

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Theorem 4a [JP] $X$ - negatively curved. For any $\delta > 0$

$$R_x(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda) \frac{P(-\mathcal{H}/2)}{h} - \delta \right), \quad n = 2, 3.$$ 

Results for $n \geq 4$ involve heat invariants.

$$K = -1 \Rightarrow R_x(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{1}{2}} - \delta \right)$$

Karnaukh, $n = 2$: estimate above + weaker estimates in variable negative curvature.
• **Global results:** $R(\lambda)$

**Randol, $n = 2$:**

$$K = -1 \Rightarrow R(\lambda) = \Omega \left( (\log \lambda)^{\frac{1}{2} - \delta} \right), \quad \forall \delta > 0.$$

**Theorem 4b [JPT]** $X$ - negatively-curved surface ($n = 2$). For any $\delta > 0$

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• **Conjecture (folklore).** On a generic negatively curved surface

$$R(\lambda) = O(\lambda^\epsilon) \quad \forall \epsilon > 0.$$

**Selberg, Hejhal:** On compact arithmetic surfaces that correspond to quaternionic lattices $R(\lambda) = \Omega \left( \frac{\sqrt{\lambda}}{\log \lambda} \right)$.

**Reason:** exponentially high multiplicities in the length spectrum; generically, $X$ has simple length spectrum.
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Proof of Theorem 4b: (about $R(\lambda)$). $X$-compact, negatively-curved surface.  

Wave trace on $X$ (even part):

$$e(t) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t).$$

Cut-off: $\chi(t, T) = (1 - \psi(t))\hat{\rho} \left( \frac{t}{T} \right)$, where

- $\rho \in \mathcal{S}(\mathbb{R})$, supp $\hat{\rho} \subset [-1, +1]$, $\rho \geq 0$, even;
- $\psi(t) \in C_0^\infty(\mathbb{R})$, $\psi(t) \equiv 1, t \in [-T_0, T_0]$, and $\psi(t) \equiv 0, |t| \geq 2T_0$.

In the sequel, $T = T(\lambda) \to \infty$ as $\lambda \to \infty$. Let

$$\kappa(\lambda, T) = \frac{1}{T} \int_{-\infty}^{\infty} e(t)\chi(t, T)\cos(\lambda t)dt$$
Key microlocal result:

Proposition 5. Let \( T = T(\lambda) \leq \epsilon \log \lambda \). Then

\[
\kappa(\lambda, T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)^\# \cos(\lambda l(\gamma)) \cdot \chi(l(\gamma), T)}{T \sqrt{|\det(I - P_\gamma)|}} + O(1)
\]

where

- \( \gamma \) - closed geodesic;
- \( l(\gamma) \) - length;
- \( l(\gamma)^\# \) - primitive period;
- \( P_\gamma \) - Poincaré map.

Long-time version of the “wave trace” formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed geodesics. Allows to isolate contribution from a growing number of closed geodesics with \( l(\gamma) \leq T(\lambda) \) to \( \kappa(\lambda, T) \) as \( \lambda, T(\lambda) \to \infty \).
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● **Proof** - separation of closed geodesics in phase space + small-scale microlocalization near closed geodesics.

● **Dynamical lemma**: Let $X$ - compact, negatively curved manifold. $\Omega(\gamma, r)$ - neighborhood of $\gamma$ in $S^*X$ of radius $r$ (cylinder). There exist constants $B > 0, a > 0$ s.t. for all closed geodesics on $X$ with $l(\gamma) \in [T - a, T]$, the neighborhoods $\Omega(\gamma, e^{-BT})$ are disjoint, provided $T > T_0$.

Radius $r = e^{-BT}$ is exponentially small in $T$, since the number of closed geodesic grows exponentially.
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Radius $r = e^{-BT}$ is exponentially small in $T$, since the number of closed geodesic grows exponentially.
• **Lemma 6.** If $R(\lambda) = o((\log \lambda)^b)$, $b > 0$ then

$$\kappa(\lambda, T) = o((\log \lambda)^b).$$

**Goal:** estimate $\kappa(\lambda, T)$ from below. Need to extract long exponential sums as the leading asymptotics of the long-time wave trace expansion.

• Consider the sum

$$S(T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)}{\sqrt{|\det(I - P_\gamma)|}}$$

• $P_\gamma$ preserves stable and unstable subspaces. Dimension 2: eigenvalues are

$$\exp \left[ \pm \int_\gamma \mathcal{H}(\gamma(s), \gamma'(s)) ds \right].$$
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$$\exp \left[ \pm \int_\gamma \mathcal{H}(\gamma(s), \gamma'(s)) ds \right].$$
• $\mathcal{P}_\gamma - Id$ is conjugate to

\[
\begin{pmatrix}
\exp \left[ \int_\gamma \mathcal{H} \right] - 1 & 0 \\
0 & \exp \left[ -\int_\gamma \mathcal{H} \right] - 1
\end{pmatrix}
\]

Thus, $S(T)$ is asymptotic to

\[
\sum_{l(\gamma) \leq T} l(\gamma) \exp \left[ -\frac{1}{2} \int_\gamma \mathcal{H} \right].
\]

Results of Parry and Pollicott $\Rightarrow$

• **Theorem 7.** As $T \to \infty$,

\[
S(T) \sim \frac{e^{P\left(-\frac{\mathcal{H}}{2}\right) \cdot T}}{P\left(-\frac{\mathcal{H}}{2}\right)}
\]

Here $P\left(-\frac{\mathcal{H}}{2}\right) \geq (n - 1)K_2/2$. 
\[ P_\gamma - \text{Id} \text{ is conjugate to } \left( \begin{array}{cc} \exp \left[ \int_\gamma \mathcal{H} \right] - 1 & 0 \\ 0 & \exp \left[ - \int_\gamma \mathcal{H} \right] - 1 \end{array} \right) \]

Thus, \( S(T) \) is asymptotic to

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Results of Parry and Pollicott ⇒

\[ \textbf{Theorem 7.} \text{ As } T \to \infty, \]

\[ S(T) \sim \frac{e^{P(-\frac{\mathcal{H}}{2}) \cdot T}}{P(-\mathcal{H}/2)} \]

Here \( P \left( -\frac{\mathcal{H}}{2} \right) \geq (n - 1)K_2/2. \)
**Dirichlet box principle** ⇒ “straighten the phases:” \( \exists \lambda \) s.t.

\[
\cos(\lambda I(\gamma)) > \nu > 0, \quad \forall \gamma : I(\gamma) \leq T.
\]

(\( \lambda I(\gamma) \) close to \( 2\pi \mathbb{Z} \)). This combined with Theorem 7 shows that \( \exists \lambda, T \) s.t.

\[
\kappa(\lambda, T) \sim \frac{\exp[P \left(-\frac{\mathcal{H}}{2}\right) T(1 - \delta/2)\]}{T}
\]

This leads to contradiction with Lemma 6. Q.E.D.

For Dirichlet principle need \( T \asymp \ln \ln \lambda \), So, get logarithmic lower bound in Theorem 4b.
Proof of Theorem 3: $N(x, y, \lambda)$

Wave kernel on $X$:

$$e(t, x, y) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t)\phi_i(x)\phi_i(y),$$

fundamental solution of the wave equation

$$(\partial^2 / \partial t^2 - \Delta)e(t, x, y) = 0, \ e(0, x, y) = \delta(x - y), \ (\partial / \partial t)e(0, x, y) = 0.$$

$$k_{\lambda,T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t)e(t, x, y)dt$$

where $\psi \in C_0^\infty([-1, 1])$, even, monotone decreasing on $[0,1]$, $\psi \geq 0$, $\psi(0) = 1$. 
Lemma 6a If $N_{x,y}(\lambda) = o(\lambda^a(\log \lambda)^b)$, where $a > 0, b > 0$ then

$$k_{\lambda,T}(x, y) = o(\lambda^a(\log \lambda)^b).$$
• **Pretrace formula.** $M$ - universal cover of $X$, no conjugate points, $E(t, x, y)$ be the wave kernel on $M$. Then for $x, y \in X$, we have

$$e(t, x, y) = \sum_{\omega \in \pi_1(X)} E(t, x, \omega y)$$

• **Hadamard Parametrix** for $E(t, x, y) \Rightarrow$

$$K_{\lambda, T}(x, y) \sim_{\lambda \to \infty} Q_1 \lambda^{\frac{n-1}{2}} \times \sum_{\omega \in \pi_1(X) : d(x, \omega y) \leq T}$$

$$\psi \left( \frac{d(x, \omega y)}{T} \right) \sin(\lambda d(x, \omega y) + \theta_n) \frac{\sqrt{Tg(x, \omega y) d(x, \omega y)^{n-1}}}{\sqrt{Tg(x, \omega y) d(x, \omega y)^{n-1}}} + O \left[ \lambda^{\frac{n-3}{2}} e^{O(T)} \right].$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3 - (n \mod 8))$, and $Q_1 \neq 0.$
Pretrace formula. $M$ - universal cover of $X$, no conjugate points, $E(t, x, y)$ be the wave kernel on $M$. Then for $x, y \in X$, we have

$$e(t, x, y) = \sum_{\omega \in \pi_1(X)} E(t, x, \omega y)$$

Hadamard Parametrix for $E(t, x, y)$ ⇒

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Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3 - (n \mod 8))$, and $Q_1 \neq 0$. 
• **Pointwise analog of the sum $S(T)$:**

$$S_{x,y}(T) = \sum_{\omega: d(x,\omega y) \leq T} \frac{1}{\sqrt{g(x,\omega y)} d(x,\omega y)^{n-1}},$$

where $g = \sqrt{\det g_{ij}}$ in normal coordinates at $x$. $S_{x,y}(T)$ grows at the same rate as $S(T)$.

• **Reason:** let $x, y \in M$, $\gamma$ - geodesic from $x$ to $y$, $\xi = (x, \gamma'(0))$, and $\text{dist}(x, y) = r$. Then

$$\sqrt{g(x, y)} r^{n-1} \ll \text{Jac}_{\text{Vert}(\xi)} G^r.$$  

Here $\text{Vert}(\xi) \in T_{\xi}SM$ - vertical subspace; $E_{\xi}^u \in T_{\xi}SM$ - unstable subspace at $\xi$.

By properties of Anosov flows, 

$$\text{Dist}[DG^r(\text{Vert}(\xi)), DG^r(E_{\xi}^u)] \leq Ce^{-\alpha r}.\text{ Therefore,}$$

$$\text{Jac}_{\text{Vert}(\xi)} G^r \ll \text{Jac}_{E_{\xi}^u} G^r = \exp\left[\int_{\gamma} \mathcal{H}\right]$$
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Our local estimates are not uniform in \( x, y \). Need Proposition 5 to prove global estimates.

**Heat trace asymptotics:**

\[
\sum_{i} e^{-\lambda_i t} \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j t^{j - \frac{n}{2}}, \quad t \to 0^+
\]

**Local:** \( \mathcal{K}(t, x, x) = \sum_{i} e^{-\lambda_i t} \phi_i^2(x) \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j - \frac{n}{2}} \),

\( a_j(x) \) - local heat invariants, \( a_j = \int_{X} a_j(x)dx \).

\( a_0(x) = 1, a_0 = \text{vol}(X) \). \( a_1(x) = \frac{\tau(x)}{6}, \tau(x) \) - scalar curvature.
"Heat kernel” estimates:

**Theorem 2b** [JP] If the scalar curvature \( \tau(x) \neq 0 \), \( \Rightarrow \) \( R_x(\lambda) = \Omega(\lambda^{n-2}) \).

**Global** [JPT] If \( \int_X \tau \neq 0 \), \( \Rightarrow \) \( R(\lambda) = \Omega(\lambda^{n-2}) \).

**Remark:** if \( \tau(x) = 0 \), let \( k = k(x) \) be the first positive number such that the \( k \)-th local heat invariant \( a_k(x) \neq 0 \). If \( n - 2k(x) > 0 \), then

\[
R_x(\lambda) = \Omega(\lambda^{n-2k(x)}).
\]

Similar result holds for \( R(\lambda) \): if \( \int a_k(x)dx \neq 0 \) and \( n - 2k > 0 \), then

\[
R(\lambda) = \Omega(\lambda^{n-2k}).
\]
• **Oscillatory error term:** subtract \([(n - 1)/2]\) terms coming from the heat trace:

\[
N_x(\lambda) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} a_j(x) \lambda^{n-2j} \left(\frac{n}{4\pi}\right)^{j/2} \Gamma\left(\frac{n}{2}-j+1\right) + R^{osc}_x(\lambda)
\]

*Warning:* not an asymptotic expansion!

Physicists: subtract the “mean smooth part” of \(N_x(\lambda)\).

• **Theorem 2c** [JP] If \(x \in X\) is not conjugate to itself along any shortest geodesic loop, then

\[
R^{osc}_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right)
\]

**Theorem 4c** [JP] \(X\) - negatively-curved. For any \(\delta > 0\)

\[
R^{osc}_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{p(-\mathcal{H}/2)}{h}} - \delta\right), \text{ any } n.
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If \(n \geq 4\) then Theorem 2b, \(R_x(\lambda) = \Omega(\lambda^{n-2})\) gives a better bound for \(R_x(\lambda)\).

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The behavior of $N(x, y, \lambda)/(\lambda^{(n-1)/2})$ was studied by Lapointe, Polterovich and Safarov. [LPS] *Average growth of the spectral function on a Riemannian manifold.* arXiv:0803.4171, to appear in Comm. PDE.
[JS] High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows. CMP 270 (2007), 813-833


Motivation: high energy asymptotics for $\Delta$ on scalars are influenced by geodesic flow $G^t$.

Question: which dynamical system influences to high energy asymptotics of the Hodge laplacian $d\delta + \delta d$, and the Dirac operator?

Answer: frame flow, or parallel transport along the geodesic flow (cf. Bolte and Glaser, Dencker, Bunke and Olbrich, [JS]). This flow was considered by V. Arnold in 1961.
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• \(k\)-frame flow: \((v_1, \ldots, v_k)\) ordered ON set of \(k\) unit vectors. \(v_1\) defines a geodesic \(\gamma\); \((v_2, \ldots, v_k)\) are parallel transported along \(\gamma\).

• It is \(\text{SO}(k-1)\)-extension of \(G^t\); ergodicity of \(m\)-frame flow \(\Rightarrow\) ergodicity of \(k\)-frame flow, \(k < m\). Dimension 2: equivalent to ergodicity of \(G^t\) (up to orientation).

• \(X\) negatively curved, \(-K_2^2 \leq K \leq -K_1^2\).

• Key object: Brin group \(B\): closure of the holonomy group around closed piecewise US-paths (segments go along stable and unstable manifolds). \(B = \text{SO}(n-1)\) \(\Rightarrow\) frame flow is ergodic and Bernoulli. Restricted holonomy \(\Rightarrow\) nonergodic frame flow.
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• The frame flow is known to be ergodic and have the $K$ property
  • if $X$ has constant curvature (Brin 76, Brin-Pesin 74);
  • for an open and dense set of negatively curved metrics (in the $C^3$ topology) (Brin 75);
  • if $n$ is odd, but not equal to 7 (Brin-Gromov 80); or if $n = 7$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicott 03);
  • if $n$ is even, but not equal to 8, and $K_1/K_2 > 0.93$, (Brin-Karcher 84); or if $n = 8$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicott 03).

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Structural stability and other properties of frame flows were studied by Pugh, Schub, Wilkinson, Policott, Burns, Dolgopyat and many others. Kaehler manifold: \( J \) is a flow invariant; full frame flow is not ergodic. Ergodicity can sometimes be proved for restricted frame flow (Brin and Gromov, 80). This implies an appropriate version of quantum ergodicity, [JSZ].