Estimates from below for the spectral function and for the remainder in Weyl's law

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- $X^n, n \geq 2$ - compact. Δ - Laplacian. Spectrum: $\Delta\phi_i + \lambda_i\phi_i = 0, \;\; 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$
  Eigenvalue counting function:
  $N(\lambda) = \#\{\sqrt{\lambda_j} \leq \lambda\}$.
  Weyl's law: $N(\lambda) = C_n V \lambda^n + R(\lambda), \; R(\lambda) = O(\lambda^{n-1})$.
  $R(\lambda)$ - **remainder.**
- **Spectral function:** Let $x, y \in X$.
  $N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y)$.
  If $x = y$, let $N_{x,y}(\lambda) := N_x(\lambda)$.
  Local Weyl's law:
  $N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y;$
  $N_x(\lambda) = C_n\lambda^n + R_x(\lambda), \qquad R_x(\lambda) = O(\lambda^{n-1})$; $R_x(\lambda)$ - **local remainder.**
- We study **lower** bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda)$.

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- Notation: $f_1(\lambda) = \Omega(f_2(\lambda)), f_2 > 0$ iff $\limsup_{\lambda\to\infty} |f_1(\lambda)|/f_2(\lambda) > 0$. Equivalently, $f_1(\lambda) \neq o(f_2(\lambda))$.
- **Theorem 1**[JP] If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

- **Theorem 2**[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

- Other results in dimension $n > 2$ involve heat invariants.

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- **Example: flat square 2-torus**
  $\lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbf{Z}$
  $\phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}, \ x = (x_1, x_2)$

  $$|\phi_j(x)| = 1 \ \Rightarrow \ N(\lambda) \equiv N_x(\lambda)$$

  Gauss circle problem: estimate $R(\lambda)$.
  Theorem 2 $\Rightarrow$ $R(\lambda) = \Omega(\sqrt{\lambda})$ - **Hardy–Landau bound.** Theorem 2 generalizes that bound for the *local* remainder.
  Soundararajan (2003):
  $$R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}(\log\lambda)^{\frac{1}{4}}(\log\log\lambda)^{\frac{3(2^{4/3}-1)}{4}}}{(\log\log\log\lambda)^{5/8}}\right).$$
- **Hardy's conjecture:** $R(\lambda) \ll \lambda^{1/2+\epsilon} \ \forall\epsilon > 0$.
  Huxley (2003): $R(\lambda) \ll \lambda^{\frac{131}{208}}(\log\lambda)^{2.26}$.

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- **Negative curvature.** Suppose sectional curvature satisfies
  $-K_1^2 \leq K(\xi, \eta) \leq -K_2^2$
  Theorem (Berard): $R_x(\lambda) = O(\lambda^{n-1}/\log\lambda)$
  Conjecture (Randol): On a negatively-curved surface, $R(\lambda) = O(\lambda^{\frac{1}{2}+\epsilon})$. Randol proved an integrated (in λ) version for $N_{x,y}(\lambda)$.
- **Theorem (Karnaukh)** On a negatively curved surface

$$R_x(\lambda) = \Omega(\sqrt{\lambda})$$

  $+$ logarithmic improvements discussed below.
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- **Thermodynamic formalism:** G^t - geodesic flow on SX, $\xi \in SX$, $T_\xi(SX) = E^s_\xi \oplus E^u_\xi \oplus E^o_\xi$,
  • $\dim E^s_\xi = n - 1$: stable subspace, exponentially contracting for G^t;
  • $\dim E^u_\xi = n - 1$: unstable subspace, exponentially contracting for G^{-t};
  • $\dim E^o_\xi = 1$: tangent subspace to G^t.
  Sinai-Ruelle-Bowen potential $\mathcal{H} : SM \to \mathbf{R}$:

$$\mathcal{H}(\xi) = \frac{d}{dt}\Big|_{t=0} \ln \det dG^t|_{E^u_\xi}$$

- **Topological pressure** $P(f)$ of a Hölder function $f : SX \to \mathbf{R}$ satisfies (Parry, Pollicott)

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- γ - geodesic of length $l(\gamma)$. $P(f)$ is defined as

$$P(f) = \sup_{\mu} \left(h_\mu + \int f d\mu \right),$$

μ is G^t-invariant, h_μ - (measure-theoretic) entropy.

- Ex 1: $P(0) = h$ - **topological entropy** of G^t. Theorem (Margulis): $\#\{\gamma : l(\gamma) \le T\} \sim e^{hT}/hT$.
Ex. 2: $P(-\mathcal{H}) = 0$.
- **Theorem 3**[JP] If X is negatively-curved then for any $\delta > 0$ and $x \neq y$

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

Here $P(-\mathcal{H}/2)/h \ge K_2/(2K_1) > 0$.

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Theorem 4a[JP] X - negatively-curved. For any $\delta > 0$

$$R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right), \; n = 2, 3.$$

Results for $n \geq 4$ involve heat invariants.

$$K = -1 \;\Rightarrow R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{1}{2} - \delta}\right)$$

Karnaukh, $n = 2$: estimate above + weaker estimates in variable negative curvature.

- **Global results:** $R(\lambda)$
  Randol, $n = 2$:

$$K = -1 \;\Rightarrow R(\lambda) = \Omega\left((\log\lambda)^{\frac{1}{2}-\delta}\right), \qquad \forall\delta > 0.$$

Theorem 4b[JPT] X - negatively-curved surface ($n = 2$). For any $\delta > 0$

$$R(\lambda) = \Omega\left((\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right).$$

- **Conjecture (folklore).** On a **generic** negatively curved surface

$$R(\lambda) = O(\lambda^{\epsilon}) \qquad \forall\epsilon > 0.$$

Selberg, Hejhal: On compact arithmetic surfaces that correspond to quaternionic lattices $R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}}{\log\lambda}\right)$.
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Proof of Theorem 4b: (about $R(\lambda)$). X-compact, negatively-curved surface.
Wave trace on X (even part):

$$e(t) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t).$$

Cut-off: $\chi(t,T) = (1-\psi(t))\hat{\rho}\left(\frac{t}{T}\right)$, where
- $\rho \in \mathcal{S}(\mathbf{R})$, $\mathrm{supp}\,\hat{\rho} \subset [-1,+1]$, $\rho \geq 0$, even;
- $\psi(t) \in C_0^\infty(\mathbf{R})$, $\psi(t) \equiv 1, t \in [-T_0, T_0]$, and $\psi(t) \equiv 0, |t| \geq 2T_0$.

In the sequel, $T = T(\lambda) \to \infty$ as $\lambda \to \infty$. Let

$$\kappa(\lambda, T) = \frac{1}{T}\int_{-\infty}^{\infty} e(t)\chi(t,T)\cos(\lambda t)dt$$

- **Key microlocal result**:
  Proposition 5. Let $T = T(\lambda) \leq \epsilon \log \lambda$. Then

$$\kappa(\lambda, T) = \sum_{l(\gamma)\leq T} \frac{l(\gamma)^{\#} \cos(\lambda l(\gamma)) \cdot \chi(l(\gamma), T)}{T\sqrt{|\det(I - \mathcal{P}_\gamma)|}} + O(1)$$

  where
  γ - closed geodesic; $l(\gamma)$ - length; $l(\gamma)^{\#}$-primitive period; $\mathcal{P}_\gamma$ - Poincaré map.

- *Long-time* version of the "wave trace" formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed geodesics. Allows to isolate contribution from a **growing number** of closed geodesics with $l(\gamma) \leq T(\lambda)$ to $\kappa(\lambda, T)$ as $\lambda, T(\lambda) \to \infty$.

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- **Proof** - separation of closed geodesics in phase space + small-scale microlocalization near closed geodesics.
- **Dynamical lemma**: Let X - compact, negatively curved manifold. $\Omega(\gamma, r)$ - neighborhood of γ in S^*X of radius r (cylinder). There exist constants $B > 0, a > 0$ s.t. for all closed geodesics on X with $l(\gamma) \in [T - a, T]$, the neighborhoods $\Omega(\gamma, e^{-BT})$ are disjoint, provided $T > T_0$.
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- **Lemma 6.** If $R(\lambda) = o((\log\lambda)^b),\ b > 0$ then

$$\kappa(\lambda, T) = o((\log\lambda)^b).$$

Goal: estimate $\kappa(\lambda, T)$ from below. Need to extract long exponential sums as the leading asymptotics of the long-time wave trace expansion.

- Consider the sum

$$S(T) = \sum_{l(\gamma)\le T} \frac{l(\gamma)}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}}$$

- $\mathcal{P}_\gamma$ preserves stable and unstable subspaces. Dimension 2: eigenvalues are $\exp\left[\pm\int_\gamma \mathcal{H}(\gamma(s), \gamma'(s))ds\right]$.

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- $\mathcal{P}_\gamma - Id$ is conjugate to

$$\begin{pmatrix} \exp\left[\int_\gamma \mathcal{H}\right] - 1 & 0 \\ 0 & \exp\left[-\int_\gamma \mathcal{H}\right] - 1 \end{pmatrix}$$

Thus, $S(T)$ is asymptotic to

$$\sum_{l(\gamma)\leq T} l(\gamma) \exp\left[-\frac{1}{2}\int_\gamma \mathcal{H}\right].$$

Results of Parry and Pollicott $\Rightarrow$

- **Theorem 7.** As $T \to \infty$,

$$S(T) \sim \frac{e^{P(-\frac{\mathcal{H}}{2})\cdot T}}{P(-\mathcal{H}/2)}$$

Here $P\left(-\frac{\mathcal{H}}{2}\right) \geq (n-1)K_2/2$.

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Dirichlet box principle $\Rightarrow$ "straighten the phases:" $\exists\lambda$ s.t.

$$\cos(\lambda l(\gamma)) > \nu > 0, \ \forall \gamma : l(\gamma) \le T.$$

$(\lambda l(\gamma)$ close to $2\pi\mathbf{Z})$. This combined with Theorem 7 shows that $\exists\lambda$, T s.t.

$$\kappa(\lambda, T) \sim \frac{\exp[P\left(-\frac{\mathcal{H}}{2}\right) T(1-\delta/2)]}{T}$$

This leads to contradiction with Lemma 6. Q.E.D.
For Dirichlet principle need $T \asymp \ln\ln\lambda$, So, get logarithmic lower bound in Theorem 4b.

Proof of Theorem 3: $N(x, y, \lambda)$
Wave kernel on X:

$$e(t, x, y) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t)\phi_i(x)\phi_i(y),$$

fundamental solution of the wave equation
$(\partial^2/\partial t^2 - \Delta)e(t, x, y) = 0,\;\; e(0, x, y) = \delta(x - y),$
$(\partial/\partial t)e(0, x, y) = 0.$

$$k_{\lambda,T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) e(t, x, y) dt$$

where $\psi \in C_0^\infty([-1, 1])$, even, monotone decreasing on [0,1], $\psi \geq 0$, $\psi(0) = 1$.

Lemma 6a If $N_{x,y}(\lambda) = o(\lambda^a (\log \lambda)^b))$, where $a > 0, b > 0$ then

$$k_{\lambda,T}(x,y) = o(\lambda^a (\log \lambda)^b)).$$

- **Pretrace formula.** M - universal cover of X, no conjugate points, $E(t,x,y)$ be the wave kernel on M. Then for $x,y \in X$, we have

$$e(t,x,y) = \sum_{\omega \in \pi_1(X)} E(t,x,\omega y)$$

- *Hadamard Parametrix* for $E(t,x,y) \Rightarrow$

$$K_{\lambda,T}(x,y) \sim_{\lambda\to\infty} Q_1 \lambda^{\frac{n-1}{2}} \times \sum_{\omega \in \pi_1(X): d(x,\omega y) \leq T}$$

$$\frac{\psi\left(\frac{d(x,\omega y)}{T}\right) \sin(\lambda d(x,\omega y) + \theta_n)}{\sqrt{Tg(x,\omega y)\, d(x,\omega y)^{n-1}}} + O\left[\lambda^{\frac{n-3}{2}} e^{O(T)}\right].$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3 - (n \bmod 8))$, and $Q_1 \neq 0$.

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Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3-(n \bmod 8))$, and $Q_1 \neq 0$.

- Pointwise analog of the sum $S(T)$:

$$S_{x,y}(T) = \sum_{\omega : d(x,\omega y) \le T} \frac{1}{\sqrt{g(x,\omega y)\, d(x,\omega y)^{n-1}}},$$

where $g = \sqrt{\det g_{ij}}$ in normal coordinates at x. $S_{x,y}(T)$ grows at the same rate as $S(T)$.

- **Reason:** let $x, y \in M$, γ - geodesic from x to y, $\xi = (x, \gamma'(0))$, and $\mathrm{dist}(x, y) = r$. Then $\sqrt{g(x, y) r^{n-1}} \ll Jac_{Vert(\xi)} G^r$.
Here $Vert(\xi) \in T_\xi SM$ - vertical subspace; $E^u_\xi \in T_\xi SM$ - unstable subspace at ξ.
By properties of Anosov flows, $\mathrm{Dist}[DG^r(Vert(\xi)), DG^r(E^u_\xi)] \le Ce^{-\alpha r}$. Therefore,
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Our **local** estimates are not uniform in x, y. Need Proposition 5 to prove **global** estimates.

Heat trace asymptotics:

$$\sum_i e^{-\lambda_i t} \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j \; t^{j-\frac{n}{2}}, \qquad t \to 0^+$$

Local: $\mathcal{K}(t,x,x) = \sum_i e^{-\lambda_i t}\phi_i^2(x) \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j-\frac{n}{2}}$,

$a_j(x)$ **- local heat invariants,** $a_j = \int_X a_j(x)dx$**.**

$a_0(x) = 1$**,** $a_0 = \mathrm{vol}(X)$**.** $a_1(x) = \frac{\tau(x)}{6}$**,** $\tau(x)$ **- scalar curvature.**

"Heat kernel" estimates:

Theorem 2b[JP] If the scalar curvature $\tau(x) \neq 0, \Longrightarrow R_x(\lambda) = \Omega(\lambda^{n-2})$.

Global:[JPT] If $\int_X \tau \neq 0, \Rightarrow R(\lambda) = \Omega(\lambda^{n-2})$.

Remark: if $\tau(x) = 0$, let $k = k(x)$ be the first positive number such that the k-th local heat invariant $a_k(x) \neq 0$. If $n - 2k(x) > 0$, then

$$R_x(\lambda) = \Omega(\lambda^{n-2k(x)}).$$

Similar result holds for $R(\lambda)$: if $\int a_k(x)dx \neq 0$ and $n - 2k > 0$, then

$$R(\lambda) = \Omega(\lambda^{n-2k}).$$

- **Oscillatory error term:** subtract $[(n-1)/2]$ terms coming from the heat trace:
  $N_x(\lambda) = \sum_{j=0}^{[\frac{n-1}{2}]} \frac{a_j(x)\lambda^{n-2j}}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-j+1\right)} + R_x^{osc}(\lambda)$
  Warning: **not** an asymptotic expansion!
  Physicists: subtract the "mean smooth part" of $N_x(\lambda)$.
- **Theorem 2c**[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x^{osc}(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

  Theorem 4c[JP] X - negatively-curved. For any $\delta > 0$
  $R_x^{osc}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}(\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right)$, *any n.*
  If $n \geq 4$ then Theorem 2b, $R_x(\lambda) = \Omega(\lambda^{n-2})$ gives a better bound for $R_x(\lambda)$.
- **Global Conjecture:** X - negatively-curved. For any $\delta > 0$
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The behavior of $N(x, y, \lambda)/(\lambda^{(n-1)/2})$ was studied by Lapointe, Polterovich and Safarov.
[LPS] *Average growth of the spectral function on a Riemannian manifold.* arXiv:0803.4171, to appear in Comm. PDE.

- [JS] High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows. CMP 270 (2007), 813-833
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- k-frame flow: $(v_1, \ldots, v_k)$ ordered ON set of k unit vectors. v_1 defines a geodesic γ; $(v_2, \ldots, v_k)$ are parallel transported along γ.
- It is $SO(k-1)$-extension of G^t; ergodicity of m-frame flow $\Rightarrow$ ergodicity of k-frame flow, $k < m$. Dimension 2: equivalent to ergodicity of G^t (up to orientation).
- X negatively-curved, $-K_2^2 \leq K \leq -K_1^2$.
- Key object: *Brin group B*: closure of the holonomy group around closed piecewise US-paths (segments go along stable and unstable manifolds). $B = SO(n-1)$ $\Rightarrow$ frame flow is ergodic and Bernoulli. Restricted holonomy $\Rightarrow$ nonergodic frame flow.

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Structural stability and other properties of frame flows were studied by Pugh, Schub, Wlkinson, Policott, Burns, Dolgopyat and many others.

Kaehler manifold: J is a flow invariant; full frame flow is not ergodic. Ergodicity can sometimes be proved for *restricted frame flow* (Brin and Gromov, 80). This implies an appropriate version of quantum ergodicity, [JSZ].