THE MANIFOLD OF METRICS WITH A FIXED VOLUME FORM

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Abstract. We study the manifold of all metrics with the fixed volume form on a compact Riemannian manifold of dimension ≥ 3. We compute the characteristic function for the L^2 (Ebin) distance to the reference metric. In the Appendix, we study Lipschitz-type distance between Riemannian metrics, and give applications to the diameter and eigenvalue functionals.

1. Introduction

The paper [CJW] initiated a program of studying the behavior of geometric properties of random Riemannian metrics; this can be thought of as performing geometric analysis on the (infinite-dimensional) manifold of Riemannian metrics on a fixed compact manifold. The authors of [CJW] took a fixed "reference" (or "background" metric g^0 on M, and considered a random metric g on an arbitrary but fixed compact Riemannian manifold M, lying in the conformal class of g^0. The conformal class was parametrized by letting the logarithm of the conformal factor vary as a Gaussian random field on M constructed using the eigenfunctions of the Laplacian for the reference metric.

In the present paper, we consider random deformations in a transverse direction: we choose a random Riemmanian metric among those having the same volume form as g^0. Again we parametrize those metrics by exponentiating a Gaussian random field on the manifold and mainly study the distribution of various distances to the reference metric. For simplicity our construction depends on a choice of an orthonormal frame in the tangent bundle (we thus make the topological assumption of *parallelizability*, that is that M supports such frames), but except for this, it is invariant under diffeomorphisms of the manifold (more precisely, the pushforward of the probability measure under a diffeomorphism would correspond to the measure obtained by pushing forward the reference metric and the frame). The construction is given in Section 3. We expect to be able to make a similar construction on general

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manifolds without the parallelizability assumption; the results of this paper should extend as well.

The construction proceeds by viewing the space of metrics with a given volume form as the space of sections of a bundle over M with fibers diffeomophic to the symmetric space $S = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ $(n = \dim M)$. This symmetric space supports an invariant Riemannian metric which can then be used to define an L^2 distance on the space of metrics. This metric coincides with the metric arising from a Riemannian structure on this (infinite-dimensional) space. This distance is introduced in Section 2.3 and is studied as a random variable in Section 4, where tail estimates are obtained in terms of geometric constants.

In the Appendix, similar constructions are carried out for a Lispchitz-type distance, also considered in [BU]. Those estimates are then applied to prove integrability results for the *diameter* and *Laplace eigenvalue* functionals of random Riemannian metrics. The authors plan to further study those and other geometric and spectral functionals in subsequent papers.

In another sequel, the authors will address questions about *convergence* and *tightness* (i.e. relative compactness in the weak-* topology) of our families of measures.

We expect that the Gaussian measure we have introduced in this paper will have applications that extend significantly beyond the basic questions we have considered here. Questions we hope to investigate in the future include the computation of correlation functions, the computation of the probability for a metric to lie in a small ball centered around the reference metric, and the behaviour of the isoperimetic constant under random deformations of the reference metric. We are quite hopeful that the explicit character of our Gaussian measure will make it a useful tool in the study of these and other questions.

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2. The space of metrics

We fix once and for all a compact smooth manifold M without boundary and write n for its dimension. We also fix a smooth volume form dv on M.

We rely crucially on the symmetric space structure of the space P of positive-definite matrices of determinant 1 and on the related structure theory of $\mathrm{SL}_n(\mathbb{R})$. In the discussion below we state the facts we use; proofs and further details may be found in the text [Ter], which concentrates on this case, and in [Hel] which develops the general theory of symmetric spaces associated to semisimple Lie groups.

2.1. **The space of metrics.** We start by giving two coordinate-free descriptions of the set of Riemannian metrics with the volume form dv on M. We then restrict to a class of manifolds for which there is a coordinate system simplifying the description.

The first description will be in terms of reductions of the frame bundle $\mathcal{F}(M)$, viewed as a principal $\mathrm{GL}_n(\mathbb{R})$-bundle over M. Recall [Stern] that for given subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$, a G-structure B_G on $\mathcal{F}(M)$ is a reduction of the bundle to that group, that is a G-invariant submanifold of $\mathcal{F}(M)$ surjecting on M such that for all $p \in B_G$ and $g \in \mathrm{GL}_n(\mathbb{R})$, we have $g \cdot p \in B_G$ if and only if $g \in G$. In this language the set of all Riemannian metrics on M is the set of $\mathrm{O}(n)$-structures on $\mathcal{F}(M)$. Now let B_G denote the $\mathrm{O}(n)$-structure on M associated to g_0 and let $\mathrm{SL}_n(\mathbb{R})$ act pointwise on the fibers of $\pi : B_G \to M$. Note that the submanifold of $\mathcal{F}(M)$ obtained by this construction corresponds to the set of orthonormal frames for the Riemannian metrics g on M having the same volume form as g_0.

We shall now give a second description which forms our main point of view. Let V be a finite-dimensional real vector space, let V^* be its dual space, and let $\mathrm{Sym}(V) = \{g \in \mathrm{Hom}(V, V^*) | g^* = g\}$ be the space of symmetric bilinear forms on V. For these we construct $\mathrm{Pos}(V) = \{g \in \mathrm{Sym}(V) | \forall v \in V : g(v,v) > 0\}$, the space of positive-definite bilinear forms on V. We also let $\mathrm{SL}(V) \subset \mathrm{GL}(V)$ denote the special (respectively general) linear group on V, and $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ their Lie algebras. Then $\mathrm{GL}(V)$ acts on $\mathrm{Pos}(V)$ by $h^{-1} \cdot g = h^* \circ g \circ h$. It is well-known that this action is transitive; the stabilizer of any $h \in \mathrm{Pos}(V)$ is a maximal compact subgroup isomorphic to $\mathrm{O}(n)$. Moreover, the orbits of $\mathrm{SL}(V)$ are precisely the level sets of the determinant function $g \mapsto \det(g_0^{-1} g)$ where g_0 is a fixed isomorphism $V \to V^*$. Each level set is then of the form $\mathrm{SL}(V)/K_{g_0}$ where $K_{g_0} = \mathrm{Stab}_{\mathrm{SL}(V)}(g_0) \simeq \mathrm{SO}(n)$ and we give it the $\mathrm{SL}(V)$-invariant Riemannian structure coming from the Killing form of $\mathrm{SL}(V)$, making it into a simply connected Riemannian manifold of non-positive curvature.

Working in local co-ordinates one can now associate to the tangent bundle TM the vector bundles $\mathrm{Hom}(TM, T^*M)$ and $\mathrm{Sym}(TM)$, the symmetric space-valued bundle $\mathrm{Pos}(TM)$, and the group bundles $\mathrm{GL}(TM)$ and $\mathrm{SL}(TM)$.

By definition, a *Riemannian metric* on M is a section of $\mathrm{Pos}(M)$; we denote the space of sections by $\mathrm{Met}(M)$. To such a metric there is an associated Riemannian volume form, and we let $\mathrm{Met}_{\mathrm{dv}}(M)$ denote the space of metrics whose volume form is dv. Fixing a metric $g_0 \in \mathrm{Met}_{\mathrm{dv}}(M)$, the above discussion identifies $\mathrm{Met}_{\mathrm{dv}}(M)$ with the space of sections of the bundle over M whose fibers are isomorphic to $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Moreover, the fibre at x of this bundle is equipped with a transitive isometric action of $\mathrm{SL}(T_x M)$, where the metric is the one pulled back from the identification with $S = \mathrm{SL}_n(R)/\mathrm{SO}(n)$ (the pullback is well-defined since the metric on S is $\mathrm{SL}_n(\mathbb{R})$-invariant).

Remark 2.1. *It is a classical result of Ebin* [Eb] *that the diffeomorphism group acts transitively on the space of smooth volume forms, and therefore that the foliation of* $\mathrm{Met}(M)$ *by the orbits of the diffeomorphism group* $\mathrm{Diff}(M)$ *descends to a foliation of* $\mathrm{Met}_{\mathrm{dv}}(M)$ *by the group* $\mathrm{Diff}_{\mathrm{dv}}(M)$ *of volume-preserving diffeomorphisms. It follows that* $\mathrm{Met}(M)/\mathrm{Diff}(M) \simeq \mathrm{Met}_{\mathrm{dv}}(M)/\mathrm{Diff}_{\mathrm{dv}}(M)$*; we regard this space as the* space of geometries *on* M.

In local co-ordinates $(x^1, \ldots, x^n)$, the above construction reads as follows. One takes the basis $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ for $T_x M$ and its dual basis $\left\{dx^i\right\}_{i=1}^n$ for $T_x^* M$. Then fibers of $\mathrm{Sym}(M)$ are represented by symmetric matrices, fibers of of $\mathrm{Pos}(M)$ by positive-definite symmetric matrices. The volume form associated to $g \in \mathrm{Met}(M)$ is then given by $|\det(g_x)|^{1/2}\, dx^1 \wedge \cdots \wedge dx^n$. $\mathrm{Met}_{\mathrm{dv}}(M)$ is then the metrics g such that $\det(g_x) = \det(g_x^0)$ for all $x \in M$, where g^0 is any metric with Riemannian volume form dv. The group $\mathrm{GL}_n(\mathbb{R})$ then acts on the fibres via (up to taking inverses) the congruence action $h \cdot g = h^t g h$, with the stabilizer of g_x being the orthogonal group $\mathrm{O}_{g_x}(\mathbb{R}) \simeq \mathrm{O}(n)$. Similarly, the group $\mathrm{SL}_n(\mathbb{R})$ acts transitively on the subset of the fibre with a given determinant, with point stabilizer $\mathrm{SO}_{g_x}(\mathbb{R}) \simeq \mathrm{SO}(n)$.

2.2. **Deforming a metric.** Fix $g^0 \in \mathrm{Met}_{\mathrm{dv}}(M)$, and let $K_x \subset G_x = \mathrm{SL}(T_x M)$ be the orthogonal group of g_x^0, which is also the stabilizer of g_x^0 under the congruence action. Pointwise, fix a frame f_x in $T_x M$ orthonormal with respect to the inner product defined by g_x^0, and let $A_x \subset G_x$ be the subgroup of matrices which are diagonal with positive entries in the basis f_x. As noted above we can idenfiy the set of positive-definite quadratic forms on $T_x M$ with the same determinant as g^0 with the symmetric space G_x/K_x.

Recall now the *Cartan* (or polar) decomposition $G_x = K_x A_x K_x$. Here if we write $h_x \in G_x$ in the form $k_{1,x} a_x k_{2,x}$, the element $a_x \in A_x$ is unique up to the action of the *Weyl group* $N_{G_x}(A_x)/Z_{G_x}(A_x)$, a group isomorphic to S_n acting by permutation of the coordinates with respect to the basis f_x. Given a_x, the two elements $k_{i,x} \in K_x$ are unique up to the fact that $Z_{K_x}(a_x)$ may not be trivial (generically this centralizer is equal to $Z_{K_x}(A_x)$, which is either trivial or $\{\pm 1\}$ depending on whether n is odd or even).

It follows that for every $g^1 \in \mathrm{Met}_{\mathrm{dv}}(M)$ we can write $g_x^1 = (k_x a_x) \cdot g_x^0$ for some $a_x \in A_x$ and $k_x \in K_x$, where a_x is unique up to the action of S_n on A_x.

Our goal is to randomly deform g^0 by elements k_x and a_x for every $x \in M$. We shall discuss the "random" aspect of the construction in the next section, and concentrate at the moment on the topological issues involved in making such constructions well-defined.

Given the orthonormal frame f_x, we can identify A_x with the space of positive diagonal matrices of determinant 1. Further, using the exponential map we may identify this group with its Lie algebra $\mathfrak{a} \simeq \mathbb{R}^{n-1}$ of diagonal matrices of trace zero. We will therefore specify a_x by choosing such a matrix at each x, that is by choosing a function $H \colon M \to \mathfrak{a}$.

While this clearly works locally, making a global identification requires a choice of frame f_x at every $x \in M$, that is an everywhere non-zero section of the frame bundle of M or equivalently a trivialization of the tangent bundle of M, something which is not possible in general. For simplicity we have decided to only discuss here the case of manifolds where such sections exist (such manfiolds are called *parallelizable*). We defer more general constructions to future papers.

Remark 2.2. *Above we required the existence of a continuous orthonormal frame. Nevertheless, parallelizability is a topological notion, independent of the choice of metric g^0. To see this note that applying pointwise the Gram–Schmidt procedure with respect to the metric g^0 to any non-zero section of the frame bundle is a smooth operation and will produce a smooth orthonormal frame.*

We survey here some facts about parallelizable manifolds, mainly to note that this class is rich enough to make our construction interesting. First, a parallelizable manifold is clearly orientable. Second, a necessary condition for parallelizability is the vanishing of the second Stiefel-Whitney class of the tangent bundle, which for orientable manifolds is equivalent to M being a *spin manifold.* Examples of parallelizable manifolds include all 3-manifolds, all Lie groups, the frame bundle of any manifold and the spheres S^n with $n \in \{1, 3, 7\}$.

2.3. **The L^2 metric.** Once the volume form is fixed, the action of $\mathrm{SL}(T_xM)$ on the stalk of $\mathrm{Met}_{\mathrm{dv}}(M)$ at x identifies it with the symmetric space $S = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. As noted above this space supports an $\mathrm{SL}_n(\mathbb{R})$-invariant Riemannian metric of non-positive curvature (we recall an explicit description below). We write its distance function d_S; we then write d_x for the well-defined metric on the stalk at x of $\mathrm{Met}_{\mathrm{dv}}(M)$. Integrating this over M then gives a metric (to be denoted Ω_2) on $\mathrm{Met}_{\mathrm{dv}}(M)$: given two Riemannian metrics $g^0, g^1 \in \mathrm{Met}_{\mathrm{dv}}(M)$ on M with the same Riemannian volume form dv, we set

$$\Omega_2^2(g^0, g^1) = \int_M d_x^2(g_x^0, g_x^1)\mathrm{dv}(x).$$

We will need to understand the metric d_S. For this, identify S with the space of positive maps $\mathbb{R}^n \to \mathbb{R}^n$, in which case we may identify all tangent spaces to S with the space $\mathrm{Sym}(\mathbb{R}^n)$ of the corresponding symmetric maps. The Riemannian metric on S is then

$$ds_g^2 = \mathrm{Tr}(g^{-1}Xg^{-1}X), \tag{1}$$

where $g \in S$ and X is a symmetric map. This metric is $\mathrm{SL}_n(\mathbb{R})$-invariant by construction; its associated distance function can be expressed using the group theory of $G = \mathrm{SL}_n(\mathbb{R})$. For this let $K = \mathrm{SO}_n(\mathbb{R})$ (a maximal compact subgroup) and let $A \subset \mathrm{SL}_n(\mathbb{R})$ be the subgroup of diagonal matrices with positive entries. We then write the Cartan decomposition (described above) as $G = KAK$, which we recall is unique up to the action by conjugation of the group of permutation matrices $S_n \subset G$ on A. Now let $gK, hK \in S = G/K$ be two positive maps of determinant 1. Then $Kh^{-1}gK$ is a well-defined element of $K\backslash G/K \simeq A/S_n$, say represented by $a \in A$. We say that g and h are in *relative position* a. It turns out ([Ter]) that the distance between gK, hK is then $d_S(gK, hK) = \|\log a\|$, where $\log a \in \mathbb{R}^n$ is the vector of logarithms of the entries of the diagonal matrix a and $\|\cdot\|$ is the usual ℓ^2 norm.

3. Gaussian measures on the space of metrics

We next turn to the question of actually constructing our Gaussian measures In view of the decomposition considered in Section 2.1, it is natural to split the construction into diagonal and orthogonal parts.

Let g^0 be our reference metric. Every other metric of $\mathrm{Met}_{\mathrm{dv}}$ is of the form $g_x^1 = k_x a_x \cdot g_x^0$ where k, a are smooth functions on M such that $k_x \in K_x$ and $a_x \in A_x$. In Sections 3.1 and 3.2 we describe random constructions of a_x and k_x respectively.

Recall that we fixed a global frame on M, which we now choose to be orthonormal wrt g^0 by Gram–Schmidt. With this frame we may identify K_x with $\mathrm{SO}_n(\mathbb{R})$ and A_x with the subgroup of diagonal matrices in $\mathrm{SL}_n(\mathbb{R})$, so that $\mathrm{Lie}(K_x)$ is identified with the Lie algebra of skew-symmetric matrices, and $\mathrm{Lie}(A_x)$ with the Lie algebra $\mathrm{diag}_0(n) \subset \mathfrak{sl}_n(\mathbb{R})$ of diagonal matrices of trace zero, isomorphic to $\mathbb{R}^{n-1}$.

For the constructions below we fix a complete orthonormal basis $\{\psi_j\}_{j=0}^{\infty} \subset L^2(M)$ such that $\Delta_{g^0}\psi_j + \lambda_j\psi_j = 0$, with λ_j being a non-decreasing ordering of the spectrum of the Laplace operator Δ_{g^0}. Our constructions are in fact independent of the choice of basis of each eigenspace, but it is more convenient to make an explcit choice.

3.1. **The radial part.** We begin by defining a measure on the space of sections H_x on M such that $H_x \in \mathrm{Lie}(A_x)$. We follow the recipe of [Mor]: choose *decay coefficients* $\beta_j = F(\lambda_j)$ where $F(t)$ is an eventually monotonically decreasing function of t and $F(t) \to 0$ as $t \to \infty$. Then set

$$H_x = \sum_{j=1}^{\infty} \pi_n(\underline{\xi}_j)\beta_j\psi_j(x), \tag{2}$$

where $\underline{\xi}_j$ are i.i.d. standard Gaussians in $\mathbb{R}^n$, and $\pi_n \colon \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection on the hyperplane $\sum_{i=1}^{n} x_i = 0$.

Finally, set

$$a_x = \exp(H_x)$$

where exp is the exponential map to A_x from its Lie algebra.

The smoothness of H defined by (2) is given by [Mor, Theorem 6.3]. The following two propositions apply whenver $\underline{\xi}_j$ in (2) denotes a d-dimensional standard Gaussian, while M has dimension n.

Proposition 3.1. *If $\beta_j = O(j^{-r})$ where $r > (q+\alpha)/n + 1/2$, then H defined by* (2) *converges a.s. in $C^{q,\alpha}(M, \mathbb{R}^d)$.*

We remark that the exponents in Proposition 3.1 are independent of d (the dimension of the "target" space). Substituting into Weyl's law, we get

Proposition 3.2. *If $\beta_j = O(\lambda_j^{-s})$ where $s > q/2 + n/4$, then H defined by* (2) *converges a.s. in $C^q(M, \mathbb{R}^d)$.*

3.2. **The angular part.** In this paper we study invariants of g^1 that can be bound only using a, and hence our later calculations only depend on the marginal distribution of a. Thus, as long as the choices of k and a are independent, the choice of k has no effect. In future work we plan to ask more detailed questions where this choice will become relevant. For example, determining the curvature of g^1 following the ideas of [CJW] requires differentiating g^1_x with respect to x and hence immediately implicates the choice of k_x. We thus propose the following specific choice, again using the recipe of Equation (2). We set

$$k_x = \exp_x(u_x)$$

where u_x is the Gaussian vector

$$u_x = \sum_{j=1}^{\infty} \underline{\eta}_j \delta_j \psi_j(x). \tag{3}$$

Here $\underline{\eta}_j \in \mathfrak{so}_n$ are i.i.d. standard Gaussian anti-symmetric matrices (i.e. each $\underline{\eta}_j$ is given by $d_n = n(n-1)/2$ i.i.d. standard Gaussian variables corresponding to the upper-triangular part of $\underline{\eta}_j$), and $\delta_j = F_2(\lambda_j)$ are decay factors, given as functions of the corresponding eigenvalues.

Proposition 3.1 above applies again to give the smoothness properties of our random sections. In particular, since the exponents in Proposition 3.1 are independent of d_n, substituting into Weyl's law we get a straightforward analogue of Proposition 3.2 for the expression (3).

3.3. **Remarks on the construction.**

Remark 3.3. *For the convenience of the reader who prefers Gaussian variables to be defined by their covariance function, we note here the covariance functions relevant to our case.*

Let $\mathfrak{g}_x = \mathfrak{sl}(T_xM)$ denote the Lie algebra of $\mathrm{SL}(T_xM)$. Then $\bigcup_x \mathfrak{g}_x$ is a vector bundle, and our Gaussian measure is defined on appropriate spaces of sections of subbundles of this bundle. With sufficient continuity it is enough to consider the covariance operator evaluated on linear functionals of the form $X \mapsto \alpha_x(X(x))$, where X is a section of the bundle and $\alpha_x \in \mathfrak{g}_x^*$.

Our Gaussian measure for the diagonal part then has the covariance functions

$$R((x,k),(x',k')) = \delta_{kk'} \sum_j \beta_j^2 \psi_j(x)\psi_j(x')\,, \tag{4}$$

where k is an index for the diagonal entries of a matrix in $\mathfrak{g}_x$, diagonal with respect to our fixed frame. The angular part has a similar covariance function.

For standard choices of β_k, we note that the covariance function for analogously-defined scalar fields would be

$$r(x,y) = \begin{cases} Z(x,y,2s) := \sum_{k=1}^{\infty} \frac{\psi_k(x)\psi_k(y)}{\lambda_k^{2s}}, & \beta_k = \lambda_k^{-s}; \\ e^*(x,y,2t) := \sum_{k=1}^{\infty} \frac{\psi_k(x)\psi_k(y)}{e^{2t\lambda_k}}, & \beta_k = e^{-t\lambda_k}. \end{cases} \tag{5}$$

In (5), $Z(x,y,2s)$ denotes the spectral zeta function of Δ_0, while $e^*(x,y,2t)$ denotes the corresponding heat kernel, both taken without the constant term *that corresponds to the constant eigenfunction ψ_0.*

Here the parameter s in $\beta_k = \lambda_k^{-s}$ determines the a.s. Sobolev regularity of the random metric g via Propositions 3.1 and 3.2. If the metric g^0 is real-analytic, then letting $\beta_k = e^{-t\lambda_k}$ makes the random metric g real-analytic as well, with the parameter t related to the a.s. radius of analyticity (the exponent in rate of decay of Fourier coefficients).

Remark 3.4. *A similar construction applies to the space of* all *Riemannian metrics on M (without necessarily fixing the volume form). We now work in the symmetric space $\mathrm{GL}(T_xM)/\mathrm{O}(g_x^0)$. The only change is that in Equation* (2) *one lets A_j be standard vector-valued Gaussians without the projection.*

There is a Riemannian structure and an L^2 metric (due to Ebin) defined on the space of all metrics. A detailed study of the metric properties of this space was undertaken in [Cl].

4. Ω_2 as a random variable

In this section we study the statistics of Ω_2^2.

4.1. **The distribution function.** We recall one definition of the (fiber-wise) distance d_x introduced in Section 2.3. For this choose a a basis for T_xM orthonormal with respect to $g^0(x)$. In this basis the reference metric g_x^0 is represented by the identity matrix, and our random metric by the matrix $g_x^1 = k_x a_x^2 k_x^{-1}$. We denote the diagonal entries of a_x $e^{b_i(x)}$. In this parametrization,

$$d_x^2(g_x^0, g_x^1) = \sum_{i=1}^{n} b_i(x)^2.$$

Accordingly,

$$\Omega_2^2(g^0, g^1) = \int_M \left(\sum_{i=1}^{n} b_i(x)^2 \right) \mathrm{dv}(x). \tag{6}$$

In our random model, the vector-valued function $b(x)$ is a Gaussian random field, chosen according to Equation (2), where here we choose π_n to be the orthogonal projection. In other words $b(x)$ is defined by projecting an isotropic Gaussian in $\mathbb{R}^n$ orthogonally to the hyperplane $\sum_i b_i(x) = 0$. Integrating over x, we find that the distribution of Ω_2^2 is given by:

$$\Omega_2^2 \stackrel{D}{=} \sum_j \beta_j^2 \sum_{i=1}^{n-1} W_{i,j}$$

where $W_{i,j} \sim \chi_1^2$ are i.i.d. (equivalently, the orthogonal projection from $\mathbb{R}^n$ to the hyperplane has the eigenvalue 1 with multiplicity $(n-1)$). We can rewrite this as

$$\Omega_2^2 \stackrel{D}{=} \sum_j \beta_j^2 V_j$$

where $V_j \sim \chi_{n-1}^2$ are i.i.d.

One can then explicitly compute the moment generating function of Ω_2^2 as the product

$$\begin{aligned} M_{\Omega_2^2}(t) = E(\exp(t\Omega_2^2)) &= \prod_j \prod_{i=1}^{n} M_{\chi_1^2}(t\beta_j^2) = \prod_j \prod_{i=1}^{n} (1-2t\beta_j^2)^{-1/2} \\ &= \prod_j (1-2t\beta_j^2)^{-(n-1)/2} \end{aligned}$$

The characteristic function can also be found explicitly as

$$\prod_j \prod_{i=1}^{n} (1-2it\beta_j^2)^{-1/2} = \prod_j (1-2it\beta_j^2)^{-(n-1)/2}.$$

4.2. **Tail estimates for Ω_2^2.** Here we apply [LM, Lemma 1, (4.1)] to estimate the probability of the following events:

$$\mathrm{Prob}\{\Omega_2^2 > R^2\}, \qquad R \to \infty. \tag{7}$$

We let $W = \sum_i a_i Z_i^2$ with Z_i i.i.d. standard Gaussians, and for $(n-1)(j-1)+1 \le i \le (n-1)j$, we have $a_i = \beta_j^2$ (i.e. each β_j^2 is repeated $(n-1)$ times). We let $\|a\|_\infty = \sup_j a_j$. Assume from now on that $\beta_j = F(\lambda_j)$ is a *monotone decreasing* function; then $\|a\|_\infty = a_1 = \beta_1^2$.

It is shown in [LM, Lemma 1, (4.1)] that for $W_k = \sum_{i=1}^{k(n-1)} a_i Z_i^2$, we have

$$\mathrm{Prob}\{W_k \ge \sum_{i=1}^{k(n-1)} a_i + 2 \left(\sum_{i=1}^{k(n-1)} a_i^2 \right)^{1/2} \sqrt{x} + 2\|a\|_\infty x\} \le e^{-x}.$$

Letting $k \to \infty$, we get the following quantities:

$W := \lim_{k\to\infty} W_k = \Omega_2^2$;
$A^2 = \sum_{i=1}^{\infty} a_i = (n-1)\sum_{j=1}^{\infty} \beta_j^2$;
$B^4 = \sum_{i=1}^{\infty} a_i^2 = (n-1)\sum_{j=1}^{\infty} \beta_j^4$;
$\|a\|_\infty = a_1 = \beta_1^2$.

We obtain

$$\mathrm{Prob}\{W \geq A^2 + 2B^2 x + 2\|a\|_\infty^2 x^2\} \leq e^{-x^2}.$$

We have to solve

$$R^2 = 2||a||_\infty^2 x^2 + 2B^2 x + A^2.$$

This gives (for $R \geq A$) the following root:

$$x(R) = \frac{-B^2 + \sqrt{B^4 + 2(R^2 - A^2)||a||_\infty^2}}{2||a||_\infty^2}. \tag{8}$$

Then we obtain

$$\mathrm{Prob}\{\Omega_2 \geq R\} \leq e^{-(x(R))^2},$$

where $x(R)$ is given by (8).

It is easy to show that there exists a constant $C = C(A, B, ||a||_\infty)$ such that for $R \geq A$, we have

$$x(R)^2 \geq \frac{R^2}{2||a||_\infty^2} - CR = \frac{R^2}{2\beta_1^2} - CR.$$

We also notice that

$$\mathrm{Prob}\{\Omega_2 \geq R\} \geq \mathrm{Prob}\{\beta_1^2 Z_1^2 \geq R^2\} = \Psi\left(\frac{R}{\beta_1}\right) \geq C\frac{\beta_1 e^{-R^2/(2\beta_1^2)}}{R},$$

provided $R \geq \beta_1$.

To summarize:

Theorem 4.1. *For $R \geq A$, we have*

$$\frac{C\beta_1}{R}\exp\left(\frac{-R^2}{2\beta_1^2}\right) \leq \mathrm{Prob}\{\Omega_2 \geq R\} \leq \exp\left(\frac{-R^2}{2\beta_1^2} + CR\right).$$

Appendix by Y. Canzani, D. Jakobson and L. Silberman
Lipschitz distance. Applications to the study of diameter and Laplace eigenvalues.

In this section we shall prove tail estimates for a Lipschitz-type distance ρ defined below, and use those estimates to prove that the *diameter* and *Laplace eigenvalue* functionals are measurable with respect to the Gaussian measures defined in Section 3, and to give tail estimates for them.

A.1. **Lipschitz distance.** Here we study a (Lipschitz-type) distance ρ related to the distance used in [BU] by Bando and Urakawa. It is defined by

$$\rho(g^0, g^1) = \sup_{x\in M} \sup_{0\neq\xi\in T_x M} \left|\ln \frac{g^1(\xi,\xi)}{g^0(\xi,\xi)}\right| \tag{9}$$

In other words, it is determined by taking the identity map on M and considering its Lipschitz constants between the two metrics.

As in the case of Ω_2, $\rho(g^1, g^0)$ depends only on a_x where $g_x^1 = k_x a_x \cdot g_x^0$. In the our adapted frame, the diagonal part of g^1 has entries $e^{2b_i(x)}$, where $\sum_i b_i(x) = 0$

for every $x \in M$, and where the vector $b(x) = (b_1(x), \dots, b_n(x))$ is defined by the formula (2). Specifically, for any $x \in$M the second supremum in (9) is equal to

$$2 \sup_i |b_i(x)| \tag{10}$$

The supremum is attained for $\xi = e_i$ (the i-th unit vector in $T_x M$). Accordingly,

$$\rho(g^0, g^1) = 2 \sup_{1 \le i \le n} \sup_{x \in M} |b_i(x)| \tag{11}$$

A.2. **Tail estimate for ρ.** Now, $\rho > R$ iff $\sup_j \sup_{x \in M} |2b_i(x)| > R$. Accordingly,

$$\mathrm{Prob}\{\rho(g^0, g^1) > R\} \le \mathrm{Prob}\{\sup_{x \in M} \sup_i |b_i(x)| > R/2\}. \tag{12}$$

Recall that $\mathrm{diag}(b_1, \dots, b_n)$ is given by projecting a random vector on a particular hyperplane, which does not increase the maximum norm. It follows that

$$\mathrm{Prob}\{\rho(g^1, g^0) > R\} \le \mathrm{Prob}\{\sup_{x \in M} \sup_j |a_j(x)| > R/2\},$$

where a_j are the components of an $\mathbb{R}^n$-valued Gaussian vector. By symmetry we have for fixed i that

$$\mathrm{Prob}\{\sup_{x \in M} |a_i(x)| > u\} \le 2 \cdot \mathrm{Prob}\{\sup_{x \in M} a_i(x) > u\}.$$

Taking the union bound we find that

$$\mathrm{Prob}\{\rho(g^0, g^1) > R\} \le 2n \cdot \mathrm{Prob}\{\sup_{x \in M} a_1(x) > R/2\}. \tag{13}$$

We would like to estimate this probability as $R \to \infty$. We will need the covariance function for the scalar random field $a_1(x)$, given by (see (4))

$$r_{a_1}(x, y) = \sum_{k=1}^{\infty} \beta_k^2 \psi_k(x) \psi_k(y),$$

where ψ_k denote the L^2-normalized eigenfunctions of $\Delta(g_0)$.

The following result now follows in a standard way from the Borell-TIS theorem; it can be easily deduced from the calculations in [CJW, §3] and [AT08, §2, (2.1.3)]. We denote by σ^2 the supremum of the variance $r_{a_1}(x, x)$:

$$\sigma^2 := \sigma(a_1)^2 := \sup_{x \in M} r_{a_1}(x, x). \tag{14}$$

Proposition A.2. *Let $\sigma(a_j)$ be as in* (14)*. Then*

$$\lim_{R \to \infty} \frac{\ln \mathrm{Prob}\{\sup_{x \in M} a_1(x) > R/2\}}{R^2} = \frac{-1}{8\sigma^2}.$$

Proposition A.2 and (13) imply the following

Corollary A.3. *Let $\sigma^2 := \sup_{x \in M} r_{a_1}(x, x)$. Then for any fixed $\epsilon > 0$*

$$\lim_{R \to \infty} \frac{\ln \mathrm{Prob}\{\rho(g_0, g_1) > R\}}{R^2} \le \frac{-1}{8\sigma^2}. \tag{15}$$

In the sequel, we shall need a slightly more precise estimate; it follows from the previous discussion and the estimates in [AT08, §2, p. 50].

Proposition A.4. *There exists $\alpha > 0$ such that for a fixed $\epsilon > 0$ and for large enough R, we have*

$$\mathrm{Prob}\{\rho(g_1, g_0) > R\} \leq 2n \exp\left(\frac{\alpha R}{2} - \frac{R^2}{8\sigma^2}\right).$$

A.3. **Diameter and eigenvalue functionals.** In this section we use Corollary A.3 to give estimates for the diameter and Laplace eigenvalues of the random metric g_1.

Lemma A.5. *Assume that* $\mathrm{dvol}(g_0) = \mathrm{dvol}(g_1)$*, and that in addition* $\rho(g_0, g_1) < R$*. Then*

$$e^{-R} \leq \frac{\mathrm{diam}(M, g_1)}{\mathrm{diam}(M, g_0)} \leq e^R. \tag{16}$$

and also

$$e^{-2R} \leq \frac{\lambda_k(\Delta(g_1))}{\lambda_k(\Delta(g_0))} \leq e^{2R}. \tag{17}$$

Proof. The definition (9) implies that for any fixed path $\gamma : [0, 1] \to M$, the ratio of its lengths with respect to the metrics g_0 and g_1 is satisfies

$$e^{-R} \leq \frac{L_{g_1}(\gamma)}{L_{g_0}(\gamma)} \leq e^R.$$

Since

$$\mathrm{diam}(M, g) = \sup_{x,y \in M} \inf_{\gamma:\gamma(0)=x,\gamma(1)=y} L_g(\gamma),$$

the inequality (16) follows.

To prove (17), we let $h \in H^1(M), h \not\equiv 0$ be a test function. Then $||h||_g^2 := \int_M h^2 \mathrm{dv}$ is independent of the metric, since the volume form dv is fixed. The Rayleigh quotient of h is equal to

$$\frac{\langle dh, dh \rangle_{g^{-1}}}{||h||_g^2},$$

where g^{-1} denotes the co-metric corresponding to g. Since the Lipschitz distance is symmetric in its two arguments, we conclude that if $\rho(g_0, g_1) < R$, then $\rho(g_0^{-1}, g_1^{-1}) < R$ as well. It follows that

$$e^{-2R} \leq \frac{\langle dh, dh \rangle_{g_0^{-1}}}{\langle dh, dh \rangle_{g_1^{-1}}} \leq e^{2R}. \tag{18}$$

By the min-max characterization of the eigenvalues (see e.g. [BU, §2]),

$$\lambda_k(\Delta(g)) = \inf_{U \subset H^1(M):\, \dim U = k+1} \sup_{h \in U,\, h \not\equiv 0} \frac{||dh||_{g^{-1}}^2}{||h||_g^2}.$$

The estimate (17) now follows from (18).

□

We next establish some integrability results for the *diameter* functional $\mathrm{diam}(M, g_1)$. They follow from Lemma A.5 and a slightly stronger form of Corollary A.3.

Theorem A.6. *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function such that for some $\delta > 0$*

$$h(e^y) = O\left(\exp\left[y^2(1/(8\sigma^2) - \delta)\right]\right).$$

Then $h(\mathrm{diam}(g_1))$ is integrable with respect to the probability measure $d\omega(g_1)$ constructed in section 3.

In the proof we shall use Proposition A.4.

Proof. Without loss of generality, assume that we have normalized g_0 so that $\mathrm{diam}(g_0) = 1$. It follows from (16) that if $\rho(g_0, g_1) < R$, then $\mathrm{diam}(g_1) \leq \mathrm{diam}(g_0) \cdot e^R = e^R$. By monotonicity, we have

$$h(\mathrm{diam}(g_1)) < h(e^R).$$

Since $h \geq 0$, the function $h(\mathrm{diam}(g_1))$ is integrable provided the sum

$$\sum_{k=N}^{\infty} h(e^k) \cdot \mathrm{Prob}\{g_1 : k-1 \leq \rho(g_1, g_0) \leq k\}$$

converges. By assumption on h and Corollary A.3, that sum is dominated by

$$2n \sum_{k=N}^{\infty} h(e^k) \exp\left(\frac{\alpha(k-1)}{2} - \frac{(k-1)^2}{8\sigma^2}\right) \leq$$

$$2n \sum_{k=N}^{\infty} \exp\left[\frac{\alpha(k-1)}{2} + \left(\frac{k^2}{8\sigma^2} - \delta k^2\right) - \frac{(k-1)^2}{8\sigma^2}\right]$$

Choosing N large enough, we find that the last sum is dominated by

$$2n \sum_{k=N}^{\infty} \exp\left[\frac{-\delta k^2}{2}\right],$$

and the last expression clearly converges. □

Remark A.7. *The proof of Theorem A.6 can be easily modified to establish analogous results for* averages *of the distance function. For example, given $t > 0$, consider the functional*

$$E_t(g) := \int_M \int_M (\mathrm{dist}_g(x, y))^t \,\mathrm{dv}(x)\,\mathrm{dv}(y).$$

We leave the details to the reader.

Another corollary is the following

Theorem A.8. *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function such that for some $\delta > 0$*

$$h(e^{2y}) = O\left(\exp\left[y^2(1/(8\sigma^2) - \delta)\right]\right).$$

Then $h(\lambda_k(\Delta(g_1)))$ is integrable with respect to the probability measure $d\omega(g_1)$ constructed in section 3.

Proof. The proof is similar to the proof of Theorem A.6. We let $\lambda_k(g_0) =: e^{2\beta_k} =: e^{2\beta}$.

It follows from (17) that if $\rho(g_0, g_1) < R$, then $\lambda_k(g_1) \leq \lambda_k(g_0) \cdot e^{2R} = e^{2(R+\beta)}$. By monotonicity of the function h, we have

$$h(\lambda_k(g_1)) < h(e^{2(R+\beta)}).$$

Since $h \geq 0$, the function $h(\lambda_k(g_1))$ is integrable provided the sum

$$\sum_{m=N}^{\infty} h(e^{2(m+\beta)}) \cdot \operatorname{Prob}\{g_1 : m-1 \leq \rho(g_1, g_0) \leq m\}$$

converges.

By the assumptions on h and Corollary A.3, that sum is dominated by

$$2n \sum_{m=N}^{\infty} h(e^{2(m+\beta)}) \exp\left(\frac{\alpha(m-1)}{2} - \frac{(m-1)^2}{8\sigma^2}\right) \leq$$

$$2n \sum_{m=N}^{\infty} \exp\left[\frac{\alpha(m-1)}{2} + \left(\frac{(m+\beta)^2}{8\sigma^2} - \delta(m+\beta)^2\right) - \frac{(m-1)^2}{8\sigma^2}\right]$$

Choosing N large enough, we find that the last sum is dominated by

$$2n \sum_{m=N}^{\infty} \exp\left[\frac{-\delta m^2}{2}\right],$$

and the last expression clearly converges.

□

Remark A.9. *Theorems A.6 and A.8 prove integrability results about the diameter and eigenvalue functionals. We plan to further study those and other functionals in future papers.*

A.4. **Volume entropy functional.** The *volume entropy* functional $h_{vol}(g)$ of a metric g was defined by Manning in [Man] as the exponential growth rate of volume in the universal cover. It was shown that for any point x in the universal cover N of a compact manifold M,

$$h_{vol} = \lim_{s \to \infty} \frac{1}{s} \ln \operatorname{vol}(B(x,s)), \tag{19}$$

where the volume and the distance in N are with respect to the metric g lifted from M; the choice does not depend on x, but *does* depend on the metric g.

We next prove the following counterpart of Lemma A.5.

Lemma A.10. *Assume that* $d\operatorname{vol}(g_0) = d\operatorname{vol}(g_1)$*, and that in addition* $\rho(g_0, g_1) < R$*. Then*

$$e^{-R} \leq \frac{h_{vol}(M, g_1)}{h_{vol}(M, g_0)} \leq e^R. \tag{20}$$

Proof. It follows from the definition of ρ that

$$B_{g_0}(x, s/(e^R)) \subset B_{g_1}(x,s) \subset B_{g_0}(x, e^R \cdot s).$$

By definition of h_{vol}, for any $\epsilon > 0$, there exists $s_0 > 0$ such that for every $s > s_0$, we have

$$h_{vol}(g_0) - \epsilon \leq \frac{1}{s} \ln \operatorname{vol} B_{g_0}(x,s) \leq h_{vol}(g_0) + \epsilon.$$

It follows that for large enough s, we have

$$\frac{1}{s} \ln \operatorname{vol} B_{g_1}(x,s) \leq e^R \frac{1}{se^R} \ln \operatorname{vol} B_{g_0}(x,s) \leq e^R(h_{vol}(g_0) + \epsilon),$$

and also that

$$e^{-R}(h_{vol}(g_0) - \epsilon) \leq e^{-R} \frac{1}{se^{-R}} \ln \operatorname{vol} B_{g_0}(x, s/e^R) \leq \frac{1}{s} \ln \operatorname{vol} B_{g_1}(x,s).$$

Letting $s \to \infty$ in those two inequalities, we conclude that

$$e^{-R}(h_{vol}(g_0) - \epsilon) \leq h_{vol}(g_1) \leq e^R(h_{vol}(g_0) + \epsilon).$$

We remark that $\epsilon > 0$ was arbitrary; this finishes the proof. □

Lemma A.10 now easily implies

Theorem A.11. *The conclusion of the Theorems A.6 remains true if the diameter functional is replaced by the volume entropy functional h_{vol}.*

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