RANK AND BOLLOBÁS-RIORDAN POLYNOMIALS: COEFFICIENT MEASURES AND ZEROS

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Abstract. We discuss some (numerical and theoretical) results about the coefficients and zeros of Tutte (dichromatic) polynomial of graphs of bounded degree whose size increases. We also discuss related results for Bollobás-Riordan polynomials.

1. Introduction

In this paper we discuss some numerical and theoretical results on the coefficients and zeros of Tutte and Bollobás-Riordan polynomials. The numerical results on the coefficients of Tutte polynomials inspired the paper [JMNT], where weak convergence of certain natural coefficient measures was investigated for sequences of bounded degree graphs that converge in the sense of Benjamini-Schramm; we generalize those results for Bollobás-Riordan polynomials in this paper. The MSc thesis [Tur] of one of the authors also included a similar result for the coefficient measure of the Tutte polynomial in the case of Benjamini-Schramm convergent sequences of planar graphs with bounded face degree. We also establish some *a priori* results for the coefficient measures of Tutte polynomials. Finally, we study (numerically) zeros of the Tutte and Bollobás-Riordan polynomials.

Below, we summarize the results in our paper.

1.1. **Tutte polynomials.**

The *Tutte (or dichromatic)* polynomial was introduced by Tutte; it is the most general graph invariant satisfying a deletion-contraction recurrence formula. After a change of variables, it can be transformed into a *rank polynomial* (or Whitney rank generating function). It contains important information about G, in particular about its connectivity properties, and about nowhere-zero flows on G. In Statistical Physics, it describes the partition function for the Potts model on G. When restricted to certain curves (or points), the dichromatic polynomial specializes to some well-known graph invariants, including chromatic polynomial, the number of spanning trees, the number of acyclic orientations etc. It is closely related to important invariants in knot theory, including the Jones polynomial. See [Wel] for an excellent survey on the properties of the Tutte polynomial.

The asymptotic behaviour of many graph invariants, including Laplace spectrum, cycle distribution, colouring properties, non-concentration of eigenvectors etc. has been studied extensively before. However, several asymptotic properties of the Tutte polynomial have not been considered before, to our knowledge. In our paper,

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we focus on the *coefficients* of the Tutte polynomial T_G (defined in (2.1)) as well as its zeroes. In our numerical experiments, we focus on random d-regular graphs G. In particular, we define a probability measure describing the concentration of the (normalized) coefficients of this polynomial in 2.4. We study these probability measures numerically, averaged over random regular graphs, in section 4.

In Section 3, we use the spanning tree expansion of T_G and results about overlap of spanning trees in G to establish Theorem 3.3, bounding the degree of T_G in terms of the size of overlap between two spanning trees of G. This result is generalized to more than two spanning trees in Lemma 3.5.

In Section 3.1, the previous results are specialized to the case of d-regular graphs G on n vertices. First, we note the following result of Catlin: a bound on the edge connectivity of G implies existence of disjoint spanning trees, Proposition 3.7. Next, we combine results due to Wormald about the edge connectivity of random regular graphs, with Proposition 3.7 and Lemma 3.5 to establish Theorem 3.8 which gives an a.s. bound on the degree of T_G for d-regular G on n vertices, as $n \to \infty$. The case of cubic graphs (regular graphs of degree 3) is considered in Theorem 3.11 about the overlap of spanning trees on such graphs. All graphs in this paper will be simple unless loops or multiple edges are explicitly allowed.

Section 4 collects the results of numerical experiments on the coefficient measure of the Tutte polynomial of random regular graphs of varying size and degree. Some comments are made regarding the shape of these distributions.

Section 5 contains various numerical representations of the zero sets of the Tutte polynomial of random regular graphs. Both *real* and *complex* zeros are considered. The real zeros lie in $\mathbb{R}^2$; the complex zeros lie in $\mathbb{C}^2$ and so we picture three-dimensional "slices" of the zero sets. Various cross sections are presented in an effort to better understand their structure. Some theoretical results are also presented.

1.2. **Bollobás-Riordan polynomials.** The Bollobás-Riordan polynomial is a 3-variable polynomial defined for *ribbon graphs* (graphs with a cyclic edge orientation at every vertex) which provides an extension of the rank polynomial to such graphs. Those polynomials were introduced in [BR01, BR02].

In the second part of our paper, we investigate asymptotic properties of the coefficients and zeros of Bollobás-Riordan polynomials for bounded degree ribbon graphs whose size increases. We define ribbon graphs in §6. Bollobás-Riordan polynomials are defined in §8. Random ribbon graphs were studied in several papers, including [Gam, Fl-P, Ch-Pit]. We summarize some of the relevant results in §9, and provide a natural extension of Benjamini-Schramm convergence for ribbon graphs. In section 10 we show that natural analogues of the coefficient measures for those graphs converge to a delta function (as the graph size increases), extending one of the main results in [JMNT]. In section 11 we include some numerical investigations of the coefficients and the zero sets of Bollobás-Riordan polynomials.

Finally, in section 12 we list some natural questions that provide further study directions.

2. Tutte polynomial

The form of the dichromatic polynomial considered in this section is the Tutte polynomial which we will denote $T_G(x, y)$. It is defined as follows: let G have n vertices and m edges. Choose an ordering of the edges of G (the result will not depend on a particular choice); for every spanning tree T of G, this ordering will define its *internal activity* $\mathrm{int}(T)$ and *external* activity $\mathrm{ext}(T)$; we refer to [Big1, Ch. 13] for details.[1] We note that $\mathrm{int}(T) \leq n-1$, and $\mathrm{ext}(T) \leq m-n+1$. Then the Tutte polynomial is defined by

$$T_G(x,y) = \sum_{i,j} t_G(i,j) x^i y^j, \tag{2.1}$$

where $t_G(i,j)$ is the number of spanning trees of G with $\mathrm{int}(T) = i$ and $\mathrm{ext}(T) = j$. We note that

$$T_G(1,1) = \sum_{i,j} t_G(i,j) = \tau(G), \tag{2.2}$$

the number of spanning trees of G.

An equivalent definition of the Tutte polynomial for connected graphs is given by ([CsFr, §2])

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{k(A)-1} (y-1)^{k(A)+|A|-n}$$

where $k(A)$ is the number of connected components of A. This is $(x-1)^{-n+1} R_G(x-1, y-1)$ where R_G is the rank polynomial of G [Big1, Ch. 13].

2.1. **Coefficient measures for the Tutte polynomial.** Below, we focus on the distribution of the coefficients of the Tutte polynomials $T_G(x,y)$.

We first discuss the question of normalization. It follows from (2.2) that $T_G(1,1) = \tau(G)$, and its behaviour for random k-regular graphs $G_{n,k}$ with n vertices was studied in [McK], where it was shown that for such graphs (assuming $k \geq 3$) we have almost surely

$$\lim_{n\to\infty} \tau(G_{n,k})^{1/n} \to \frac{(k-1)^{k-1}}{(k^2-2k)^{k/2-1}}. \tag{2.3}$$

Given a connected graph G with n vertices and m edges, we associate to each graph a probability measure μ_G in the unit square $[0,1] \times [0,1]$ associated to the coefficients of the Tutte polynomial $T_G(x,y)$ as follows:

$$\mu(G) \ := \ \frac{1}{\tau(G)} \sum_{i,j} \tau_G(i,j) \cdot \delta(i/m, j/m), \tag{2.4}$$

where δ denotes the Dirac delta-function. It follows from (2.2) that $\mu(G)$ is indeed a probability measure. Also, it follows from earlier remarks that for any spanning tree T of G we have

$$\mathrm{int}(T) + \mathrm{ext}(T) \leq (n-1) + (m-n+1) = m.$$

It follows that in (2.1), for any monomial we have $(i+j)/m \leq 1$, and hence in (2.4) $\mu(G)$ is actually supported in the triangle Δ with vertices at $(0,0), (1,0)$ and $(0,1)$.

[1] Order the edges of G in an arbitrary way. An edge $e \notin T$ is *externally active* if it is minimal in its fundamental cycle, which is formed by the union of e and the path connecting the endpoints of e inside T; $\mathrm{ext}(T)$ is equal to the number of externally active edges in $E(G) \setminus T$. An edge $e \in T$ is called *internally active* if it is minimal in its fundamental cocycle, which is formed by the union of e and all the edges in $E(G) \setminus T$ having exactly one endpoint in each of the two subtrees obtained from T by removing e; $\mathrm{int}(T)$ is equal to the number of internally active edges in T.

2.2. **Sequences of graphs.** Consider a sequence G_i of graphs with n_i vertices and m_i edges. By weak compactness of the space of probability measures on Δ, we know that the sequence of probability measures $\mu(G_i)$ will have convergent subsequences. We would like to understand the limit measures of those sequences.

We concentrate on sequences of regular graphs $G_{n,k}$, but remark that in principle this question can be studied for arbitrary sequences.

2.3. **Previous work.** Limits of Tutte polynomials for recursive families of graphs ([BDS]) have been studied in [CS]; several other families were considered in [Man] and [LFH]. Weak convergence of measures on the roots of Tutte polynomials with one fixed variable was found by [CsFr]

3. Distribution of the coefficients: a priori results

Below, we establish some *a priori* results about the coefficients of the Tutte polynomial T_G that are of independent interest.

We first remark that the constant term of the Tutte polynomial is always 0. Indeed, since there is a finite number of edges, there must be a first one in the arbitrary ordering. Call it e. Let T be any spanning tree. If $e \in T$, then clearly e is the smallest bridge between the two subsets disconnected by removing it from T. Similarly, if $e \notin T$, then e must be the smallest edge in $\mathrm{Cyc}(T, e)$ (a cycle formed by the union of e and the path connecting the endpoints of e in T). Thus e must always be either internally or externally active and so either x or y must have exponent at least 1 in any spanning tree.

We next define the concept of maximal and minimal spanning trees. We number the edges by their ordering and assign each a length equal to its number. Then construct the spanning trees that maximize and minimize the total lengths and denote them $T_{\min}$ and $T_{\max}$ respectively. This leads to the following two Lemmas; they are probably standard, but we include the proofs for completeness.

Lemma 3.1. *Let T be a spanning tree and let e be internally active with respect to T. Then $e \in T \cap T_{\min}$.*

Proof. Removing e from T creates two disconnected trees spanning sets of vertices A and B. Since e is internally active, it is the smallest bridge between A and B (an edge with one endpoint in A and another endpoint in B). Suppose $e \notin T_{\min}$. Adding it to $T_{\min}$ will create a cycle $\mathrm{Cyc}(T_{\min}, e)$. This cycle includes vertices in both A and B and therefore contains at least two bridges between the two sets. Let f be the smallest bridge other than e. Now, since e is the smallest bridge between A and B, $e < f$. Thus replacing f by e in $T_{\min}$ reduces the total length, contradicting the definition. Hence, e must have been in $T_{\min}$ and so in $T \cap T_{\min}$. □

Let T be a spanning tree of G; we denote its complement by by $T^{\complement}$.

Lemma 3.2. *Let T be any spanning tree and let e be externally active with respect to T. Then $e \in T^{\complement} \cap T^{\complement}_{\max}$.*

Proof. Let e be externally active and suppose $e \in T_{\max}$. Remove e from $T_{\max}$ to obtain two trees spanning sets A and B. Now consider $\mathrm{Cyc}(T, e)$. It must contain at least one other bridge f between A and B. Now, by definition, e is the smallest edge in $\mathrm{Cyc}(T, e)$ so $e < f$. Hence replacing e with f in $T_{\max}$ increases its total length. This contradicts the definition of $T_{\max}$ and so $e \notin T_{\max}$, proving our theorem. □

It is important to note that the previous two lemmas are not if and only if statements. They only specify sets that must contain all the internally or externally active edges, not the set of such edges. They lead us to the following theorem.

Theorem 3.3. *Let G be a graph with n vertices and m edges. Suppose also it has two spanning trees T_1 and T_2 that overlap in only k edges. Suppose also that this overlap is minimal. Then the Tutte polynomial of G cannot have terms of degree more than $m-(n-k-1)$.*

Proof. Let T_1 and T_2 be the two trees overlapping on only k edges. The Tutte polynomial is independent of the ordering so we may select a specific ordering. Set T_1 to be $T_{\min}$ by numbering its edges first. Let the k overlapping edges be $n-k$ to $n-1$. We now want T_2 to be $T_{\max}$. Assign the largest $m-1-k$ numbers to the remaining edges of T_2, and the remaining numbers to the edges of $(T_1 \cup T_2)^{\complement}$ in any way. We claim that T_2 is now $T_{\max}$.

Let T be any spanning tree other than T_2. We need to show that its total length is less than that of T_2. Since, by hypothesis, any two spanning trees have overlap at least k, $T \cap T_{\min}$ must contain at least k edges. Since both T and T_2 have $n-1$ edges, we define a bijection f from T_2 to T as follows.

(1) f is the identity on $T \cap T_2$.
(2) f maps $T_2 \cap T_{\min}$ into $T_{\min} \cap T$. This is possible since $T_2 \cap T_{\min}$ has k elements and $T_{\min} \cap T$ contains at least that many.

We now claim that f is a non-increasing function when the edges are ordered by length. Denote the length function by L. Let e be an edge in T_2. If $e \in T$, then $f(e) = e$ and so $L(f(e)) = L(e)$. If not and $e \in T_{\min} \cap T_2$, then $L(e) \geq n-k$ given how the edges were numbered in $T_{\min}$. Furthermore, since $e \notin T \cap T_2$, $f(e) \notin T \cap T_2$. Now apply condition (2) and we have that $f(e) \in T_{\min}$ but not in T_2. Hence, by our numbering, $L(f(e)) < n-k$ and so $L(f(e)) < L(e)$. The last case is if $e \notin T_{\min}$ and $e \notin T$. By the first statement and our numbering, $L(e) \geq m-k$. By the second statement, $f(e) \notin T_2$ meaning that $L(f(e)) < m-k$. Hence once again we have $L(f(e)) < L(e)$.

Thus, since f is constant on the overlap and strictly decreasing on $T_2 \setminus (T \cap T_2)$, the total length of T is less than that of T_2 for any spanning tree T which is not identically T_2. Hence $T_2 = T_{\max}$.

Now, by the previous two lemmas, an edge can only be active (externally or internally) if it is in $S = T_{\min} \cup T_{\max}^{\complement}$. Since $T_{\max}$ contains $n-1-k$ elements not in $T_{\min}$, S has size $m-(n-1-k)$ and so the maximum degree of a term in the Tutte polynomial is $m-(n-1-k)$. □

Remark 3.4. In the special case where two non-overlapping spanning trees exist, the total activity cannot exceed $m+1-n$.

Some graphs have many entirely non-overlapping spanning trees. The following theorem gives some bounds on activity of certain spanning subtrees of these graphs.

Lemma 3.5. *Let G be a graph with n nodes and m edges having k non-overlapping spanning trees numbered $T_1, T_2, ..., T_k$. Let T be a spanning tree that does not overlap with the last ℓ trees of this list for some $\ell < k-1$. Then the contribution of T to the Tutte polynomial of G cannot have degree more than $m-(\ell+1)(n-1)$.*

Proof. Number the edges of T_1 with the numbers 1 through $n-1$. Number the edges in the trees T_2 to T_k with the numbers $m-k(n-1)+1$ through m, with the edges in T_i having smaller numbers than the edges in T_j if $i < j$. Number the remaining edges with the numbers in between.

Now define $G^* = G \setminus T_{\max,G}$. Recall that T_{min}, T_{max} were defined just before Lemma 3.1. Let G^{p*} denote G after p applications of the $*$ operator. By our choice of numbering $T_1 = T_{\min,G}$, $T_k = T_{\max,G}$, $T_{k-1} = T_{\max,G^*}$ and so on. Furthermore, since edges in $T_{\max}$ cannot be externally active, the activity of any spanning subtree T contained in G^* will be the same in G as in G^* provided the maximal and minimal trees have no overlap.

Now choose a spanning tree T satisfying the conditions of the theorem for some $\ell < k-1$. Then $T \subseteq G^{\ell *}$. Each of these applications of $*$ removes $n-1$ edges from G meaning that $G^{\ell *}$ has $m - \ell(n-1)$ edges. Since $\ell < k-1$, there are at least two non-overlapping spanning trees so, by theorem 3.4, the total activity of T is at most $m - \ell(n-1) - (n-1) = m - (\ell+1)(n-1)$. □

The previous results concerned the bounds on the external and internal activity of specific trees. The following theorem give bounds on the average activity over all spanning trees of a graph. For what follows let any order be given on the edges and denote it by $\leq$. Internal and external activity will be defined with respect to $\leq$. Reverse internal activity and reverse external activity will be defined to be internal and external activity with respect to the reverse order $\geq$.

The following result may be standard, but we include the proof for completeness.

Theorem 3.6. *Let G be a simple graph on n vertices and m edges with b bridges.*

(1) *The average external activity over all spanning trees of G is at most $(m - n + 1)/2$.*
(2) *The average internal activity over all spanning trees of G is at most $(n - 1 + b)/2$.*

Proof. For (1), let T be a spanning tree of G and consider an edge e not in T. Let C denote the unique cycle in $T \cup e$. C must contain at least 3 edges and so e cannot be both the largest and the smallest edge in C. Thus e cannot be both externally active and reverse externally active. Hence the sum of the external and reverse external activity of any spanning tree cannot exceed $m - n + 1$. Thus the average of this quantity over all spanning trees is at most $m-n+1$. However, since the Tutte polynomial does not depend on the ordering of the edges, the average external activity and the average reverse external activity are equal. Thus twice the average external activity is at most $m-n+1$ and the average external activity is at most $(m - n + 1)/2$.

The proof of (2) is similar. Let T be a spanning tree of G and let e be an edge in T. Let C be the set of edges in G joining the two components of $T - e$. If e is a bridge, it is the only element of C and thus it is both internally and reverse internally active. If e is not a bridge, C contains at least two elements and so e cannot be both internally and reverse internally active. Thus the sum of the internal and reverse internal activity of T is at most $n - 1 + b$. Proceeding as before, the average internal activity is at most $(n - 1 + b)/2$. □

3.1. **Applications to regular graphs.** We next apply the results from Section 3 to families of regular graphs. Theorem 3.3 implies that in a graph G with n vertices, $m \geq 2n - 2$ edges, and at least two edge-disjoint spanning trees, the degree of the Tutte polynomial is at most $m - n + 1$. So, finding edge-disjoint spanning trees in G would imply a bound on the degree of $T_G(x, y)$.

In the graph theory literature, the maximal number of edge-disjoint spanning trees in G is called a *tree packing number* of G and is often denoted by $\sigma(G)$. Many interesting results about $\sigma(G)$ can be found in a survey [Pal] and references therein. The following basic observation is due to Catlin, see [Cat] or [Pal, Cor. 4]:

Proposition 3.7. *If the edge connectivity $\lambda(G)$ of the graph G satisfies $\lambda(G) \geq 2k$, then $\sigma(G) \geq k$, i.e. G has k edge-disjoint spanning trees.*

It is known (see [Wor]) that for random d-regular graphs $G_{n,d}$ on n vertices, $\lambda(G_{n,d}) = d$ asymptotically almost surely, as $n \to \infty$. It follows that $\sigma(G_{n,d}) = \lfloor d/2 \rfloor$. If $d \geq 4$, then we have $\lfloor d/2 \rfloor \geq 2$, and so Theorem 3.3 applies.

We remark that an d-regular graph on n vertices has $m = dn/2$ edges.

Theorem 3.8. *Let $d \geq 4$. Then the degree of the Tutte polynomial of a random d-regular graph $G \in G_{n,d}$ on n vertices satisfies $\deg T_G \leq dn/2 - n + 1$ almost surely as $n \to \infty$.*

This explains the "coefficient-free strips" in the figures in Section 4

For cubic graphs (regular graphs of degree 3), $\lfloor d/2 \rfloor = 1$, so this case requires a separate consideration. We first consider the case of simple graphs. The depth-first search algorithm and the construction of the tree are first discussed.

Algorithm 1. Let G be a finite graph. Let S be an empty stack. Let T, the depth-first search tree begin empty.

(1) Choose any vertex v of G and add it to T.
(2) Push each of the neighbours of v onto S, together with the edge connecting it to v.
(3) Pop S and call the vertex v. Call its associated edge e.
(4) If v is not already in T, add v and e to T.
(5) Repeat steps 2 through 4 until T is a spanning tree.

The tree described above T is called the depth-first search tree. The method for choosing the first vertex and the order in which neighbours are pushed to the stack are not described. Some authors add vertices to the tree as they are pushed to the stack, rather than after they have been popped. This is called the preorder depth first search tree and behaves quite differently. What is described in the above algorithm and will be used in what follows is sometimes referred to as the postorder depth first search tree.

Lemma 3.9. *The complement of a depth-first search tree in a cubic graph G without loops or multiple edges is acyclic.*

Proof. Choose a vertex v_0 and perform a depth-first search starting at v_0, building the tree T as we go. Arguing for a contradiction, suppose that C is a cycle contained in $G \setminus E(T)$. Since G has no loops or multiple edges, C must be incident on at least three vertices. Call this set of vertices V_C. Each of these vertices has two edges in C and one edge in the complement of C. Hence $V_C \neq V$.

Suppose $v_0 \notin V_C$. Let v be the first vertex in V_C discovered by the search. It must have been reached via its edge not in C. Now consider the two remaining edges incident on v. Both are in C and lead to vertices that have not been discovered since v is the first. Thus, following the next edge incident on v does not create a cycle and so that edge is added to T hence $C \not\subseteq G \setminus T$.

Now suppose $v_0 \in V_C$. Then let v be the second vertex in V_C to be discovered by the search. If v was reached via an edge in C, then we are done since this edge is in T. Otherwise, consider either of the other two edges incident to v. It is in C. If it does not lead to v_0, it does not create a cycle and is added to T. If it leads to v_0, it is not added to T and we move to the third edge incident on v. This edge is also in C and leads to an undiscovered vertex. So, either way, we find an edge in C to add to T. □

The following lemma is a standard result in matroid theory. We include the proof for completeness.

Lemma 3.10. *Any acyclic subset of edges S of a connected graph G can be extended to a spanning tree.*

Proof. This can be done with a greedy algorithm. The subgraph S has k connected components. Since G is connected, there must be an edge in $G \setminus S$ connecting two of these components. Add it to S. It cannot create a cycle. There are now $k-1$ connected components in S. Repeat the algorithm until there is only one. The set S is then a spanning tree. □

Theorem 3.11. *Any cubic graph G on n vertices without loops or multiple edges has two spanning trees T_1 and T_2 such that $T_1 \cup T_2 = G$. In particular, $\#(T_1 \cap T_2) = n/2 - 2$. Thus the degree of the Tutte polynomial for cubic simple graphs is at most $3n/2 - n + 1 + (2n - 2 - 3n/2) = n - 1$.*

Proof. Choose a vertex v_0 and let T_1 be a depth-first search tree starting there. Let T_2 be a spanning tree obtained by extending $G \setminus T_1$. Then $T_1 \cup T_2 = G$ since $G \setminus T_1 \subseteq T_2$. Thus, by a counting argument, the spanning trees are minimally overlapping. □

Next consider the case where we allow multiple edges, but not loops. Lemma 3.9 can be extended to cover these cases provided there is at least one vertex without multiple edges. In this case, set v_0 to be such a vertex. For any cycle C, which can be incident on as few as 2 vertices, let v be the first vertex in V_C reached by the depth-first search. It must be reached by an edge not in C and so the remaining two edges incident on v must be in C and must therefore lead to undiscovered edges. Thus one of them is added to T and so C is not in $G \setminus T$.

Triple edges are impossible in connected cubic graphs of more than 2 vertices. If every vertex has a double edge, then we must have one large ring with double edges every second link. A depth first search tree on this kind of graph will have an acyclic complement provided its first edge added in the depth first search is one of a double edge.

Thus, even allowing multiple edges, the degree of Tutte polynomial of a cubic graph on n vertices without loops is at most $n - 1$.

In the case where there are loops in the graph, the loops cannot be part of any spanning tree. By a similar reasoning to that above, the complement of a depth first search is acyclic once we remove the loops. Hence, in a cubic graph with ℓ loops, n vertices and no multiple edges, the size of the minimal overlap is $2n-2-(3n/2-\ell)$ and so the degree of the Tutte polynomial is at most $n - 1 + \ell$.

3.2. **Another argument in the case of cubic graphs.** In this section, we use the second definition of the Tutte polynomial to give an upper bound on the degree of the Tutte polynomial of a simple random cubic graph. Essentially, we want to bound

$$2k(A) + |A| + n - 1$$

First of all, we rewrite this as

$$2k(A) - \left|A^{\mathtt{C}}\right| - n + m - 1 \leq \sum_{C \text{ component of } A} (2 - out(C) - |C|) + m - 1$$

where $out(C)$ is the number of edges joining C and $C^{\mathtt{C}}$ in E. Now we claim that $2 - out(C) - |C| \leq - |C|/2$ for all components C.

First of all, if $|C| \geq 4$, $2 - |C| \leq |C|/2$ so the claim holds. For $|C| \leq 3$, we check the cases one at a time. If a C contains only one vertex, $out(C) = 3$ since the graph is cubic. Thus $2 - out(C) - |C| = 2 - 4 = -2 \leq -1/2$. If $|C| = 2$ then we must have one edge joining the two vertices leaving $out(C) = 4$. Thus $2 - out(C) - |C| = 2 - 4 - 2 = -4 \leq -1$. If $|C| = 3$, then there are at most three edges inside C. Thus $out(C) \geq 3$. Hence $2 - out(C) - |C| \leq 2 - 3 - 3 = -4 \leq -3/2$. Thus in all cases the claim is satisfied.

Thus, since every vertex is in a component, $\sum_{C \text{ component of } A} (2 - out(C) - |C|) \leq -n/2$. Finally, since in a cubic graph on n vertices, $m = 3n/2$, this gives us that

$$2k(A) - \left|A^{\mathtt{C}}\right| - n + m - 1 \leq -\frac{n}{2} + \frac{3n}{2} - 1 = n - 1$$

Thus the maximum degree of the Tutte polynomial of a simple cubic graph on n vertices is $n - 1$.

4. Numerical experiments on the coefficient measure

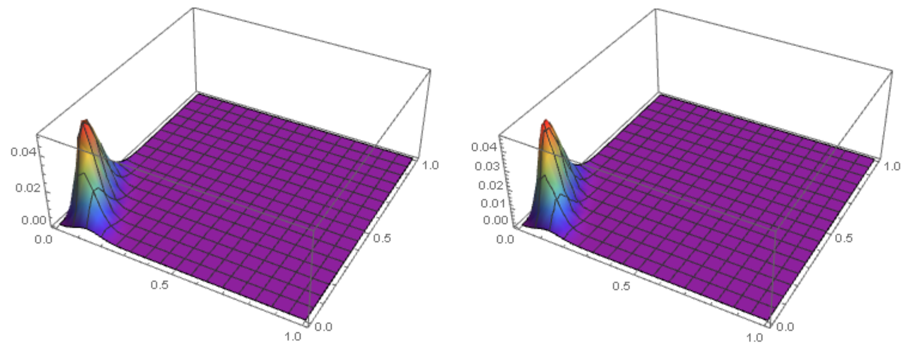
In this section, we present some numerical experiments on the coefficient measure. These were obtained by uniformly sampling random regular graphs, computing the coefficient measure of each and then averaging over 100 graphs.

It was very convenient for us to use Mathematica, since it conveniently has built-in command `RandomGraph[DegreeDistribution]`, sampling random graphs with a given degree distribution.

In addition, Mathematica has a built-in command `TuttePolynomial[G,{x,y}]` which produces the Tutte polynomial of the graph G. The standard Mathematica commands were then used to plot the zeros and the coefficients.

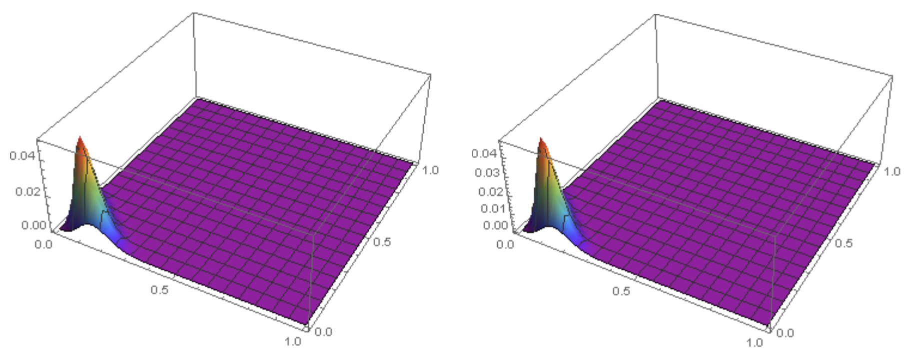
In Figure 1 below, the distribution of the coefficients of T_G was plotted, averaged over 100 graphs as follows: (A) 3-regular graphs with 22 vertices; (B) 3-regular graphs with 24 vertices.

In Figure 2 below, the distribution of the coefficients of T_G was plotted, averaged over 100 graphs as follows: (A) 4-regular graphs with 19 vertices; (B) 4-regular graphs with 20 vertices; (C) 5-regular graphs with 16 vertices; (D) 6-regular graphs with 17 vertices.

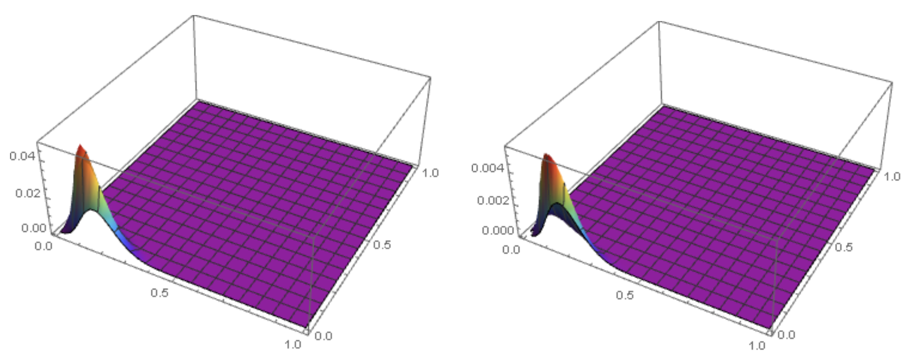


(A) (B)

FIGURE 1. Coefficients of the Tutte polynomial: 3-regular graphs



(A) (B)

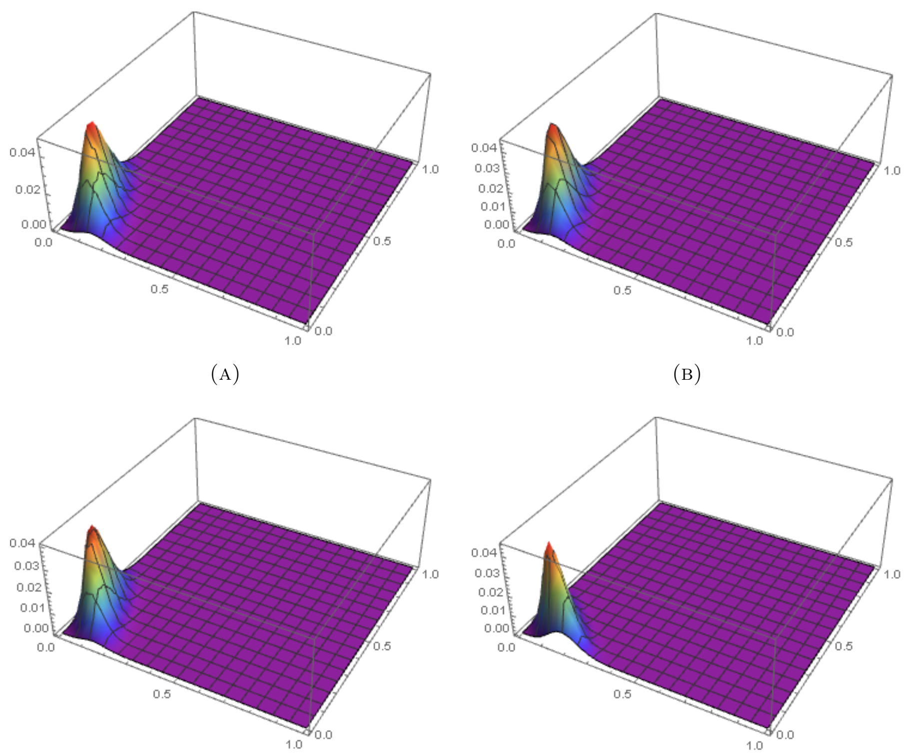


(C) (D)

FIGURE 2. Coefficients of the Tutte polynomial: 4- 5- and 6-regular graphs

To generate random planar graphs, we used `RandomGraph` command inMATHEMATICA to generate random regular graphs, then used `PlanarGraphQ[G]` command to check for planarity, and discarded the graphs that were not planar. The following coefficient distributions are obtained by sampling only planar graphs.

In Figure 3 below, the distribution of the coefficients of T_G was plotted, averaged over 100 planar graphs as follows: (A) 3-regular planar graphs with 20 vertices; (B) 3-regular planar graphs with 22 vertices; (C) 3-regular planar graphs with 24 vertices; (D) 4-regular planar graphs with 19 vertices.



(A) (B) (C) (D)

FIGURE 3. Coefficients of the Tutte polynomial: planar graphs

We end this section by plotting the distribution of the Tutte coefficients coefficients of the Petersen graph in Figure 4:

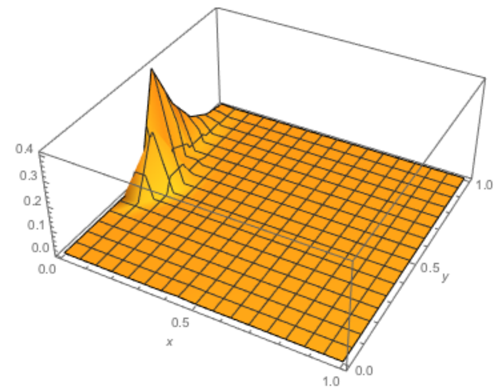


FIGURE 4. Tutte coefficients: the Petersen graph

4.1. **Asymptotic behaviour of the coefficients.** It appears, based on the numerical experiments described above, that the coefficient measures μ_G, averaged over regular graphs on n vertices, converge to a delta function as $n \to \infty$. The coefficient measures for planar graphs seem to have a similar behaviour. A related result for the coefficient measures of *rank polynomials* were established in the paper [JMNT], and in the MSc thesis [Tur] of one of the authors. The authors plan to further study questions about the coefficients and other properties of the graph polynomials in the near future. Some of those questions are discussed in Section 12.

5. ZEROS OF TUTTE POLYNOMIALS

Although the main focus of this paper is the coefficient measure of the Tutte polynomial, the *zero sets* $\mathcal{N}(T_G)$ of T_G are also of interest. This section presents a brief overview of results in this area. It concludes with some numerical results illustrating various cross sections of the zero sets. The convergence of roots of $T_G(x, y)$ *for fixed* y was addressed in several papers, including [Sok], [CsFr] and many others. Here we would like to study the behaviour of $\mathcal{N}(T_G)$ as a subset of $\mathbb{R}^2$ (or of $\mathbb{C}^2$).

Often the following form of the Tutte polynomial is considered in the literature:

$$Z_G(q, v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|}, \tag{5.1}$$

where the sum is taken over all subsets A of the edge set E of G, and $k(A)$ denotes the number of connected components of the graph (V, A). The two forms are related (see e.g. the Introduction in [Sok]) by

$$T_G(x, y) = (x - 1)^{-k(E)}(y - 1)^{-|V|} Z_G((x - 1)(y - 1), y - 1);$$
$$Z_G(q, v) = (q/v)^{k(E)} v^{|V|} T_G(1 + q/v, 1 + v),$$

so their zero sets are equivalent. In statistical physics, limits of zeros of $Z_G(q, v)$ for fixed q correspond to phase transitions in the q-state Potts model (cf. the Introduction in [Sok] and references therein). Many result surveyed below were established for the zero set $\mathcal{N}(Z_G)$.

Below, we shall restrict ourselves to connected graphs $G = (V, E)$. We first discuss some heuristics about the behaviour of the nodal set $\mathcal{N}(Z_G)$ as $|q|, |v| \to \infty$. That behaviour of $\mathcal{N}(Z_G)$ is determined by the highest powers of q and v in Z_G. Clearly, for any $A \subseteq G$, we have $k(A) \leq |V|$, with equality iff $A = \emptyset$. Accordingly, $Z_G(q, v) = q^{|V|} + terms\ of\ lower\ degree\ in\ q$. Similarly, for any $A \subseteq G$, we have $|A| \leq |E|$, with equality iff $A = E$. Accordingly, $Z_G(q, v) = v^{|E|}q + terms\ of\ lower\ degree\ in\ v$. Keeping just those highest degree terms of Z_G , we get $q^{|V|} + qv^{|E|} + lower\ order$. The zero set of the highest degree terms (after division by q) is $v^{|E|} + q^{|V|-1} = 0$. For d-regular graphs, we have $|V| = n$ and $|E| = dn/2$, so the previous equation becomes $q^{n-1} + v^{nd/2} = 0$. It seems interesting to see how accurately this very naive expression approximates the real nodal set of Z_G in the limit $|q|, |v| \to \infty$; we hope to address that question in a future project.

An important question is to find explicit bounds for $\mathcal{N}(Z_G)$ and to see how they depend on a graph G. In the paper [JPS, Theorems 1.2 and 1.3], the authors proved that for simple graphs, if v is fixed, then all zeros q of $Z_G(q, v)$ lie in the disc

$$|q| < K_\mu^* \Delta^*(G, v),$$

where $K_\mu^* \leq 5 + 2\mu, \mu = \widehat{\Delta}(G, v)/\Delta^*(G, v)$ and where $\Delta^*(G, v)$ and $\widehat{\Delta}(G, v)$ are given by the following expressions:

$$\Delta^*(G, v) = \max_{x \in V} \sum_{x \in e, e=(xy)} \min\{|v|, \frac{|v|}{\sqrt{|1+v|}}\} \prod_{y \in f} \max\{1, |1+v|\}^{1/2};$$

and

$$\widehat{\Delta}(G, v) = \max_{x \in V} \sum_{x \in e, e=(xy)} \min\{|v|, \frac{|v|}{|1+v|}\} \prod_{y \in f} \max\{1, |1+v|\}^{1/2}.$$

We remark that we have specialized the formulas in [JPS] (valid for multivariate Tutte polynomials) to the case where the edge weights $w_e = v$ for every edge $e \in E$. The results in [JPS] generalized earlier results of Sokal [Sok].

In the special case of d-regular graphs $G_{n,d}$, we have

$$\Delta^*(G, v) = d \min\{|v|, \frac{|v|}{\sqrt{|1+v|}}\} \max\{1, |1+v|\}^{d/2},$$

and

$$\widehat{\Delta}(G, v) = d \min\{|v|, \frac{|v|}{|1+v|}\} \max\{1, |1+v|\}^{d/2};$$

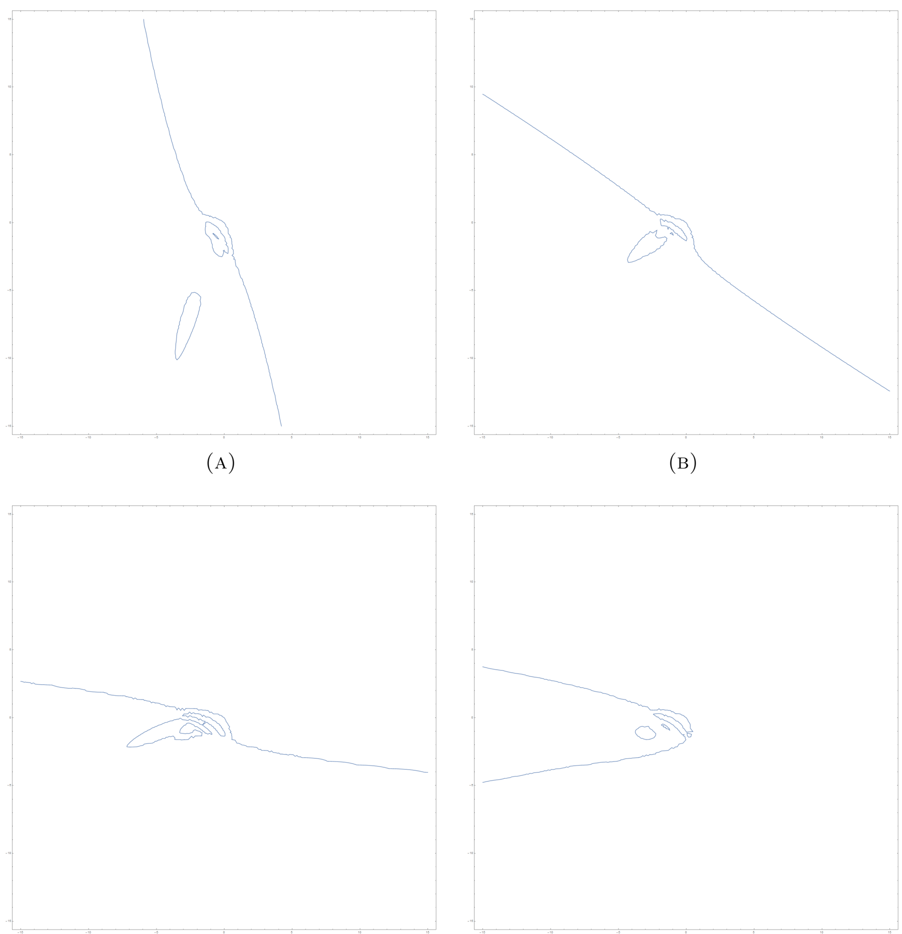
therefore

$$\mu = \frac{\min\{1, |1+v|^{-1}\}}{\min\{1, |1+v|^{-1/2}\}}.$$

5.1. **Zeros: experimental results.** In this section, we looked at the zeros of $T_G(x, y)$ for one random regular graph at a time (without averaging). We first considered real zeros. As $|x|, |y| \to \infty$, the zero set seems to be asymptotic to an algebraic curve; we do prove any rigorous results in that direction, but hope to address this question in future work.

We used standard `Mathematica` commands to plot zeros of Tutte polynomials.

In Figure 5 below, real zeros of T_G were plotted, for the following random regular graphs: (A) a 3-regular graph with 16 vertices; (B) a 4-regular graph with 16 vertices; (C) a 6-regular graph with 16 vertices; (D) a 5-regular graph with 10 vertices.



(A) (B) (C) (D)

FIGURE 5. Real Tutte zeros: random regular graphs

We next plot the real zeros of the Tutte polynomial of the Petersen graph in Figure 6.

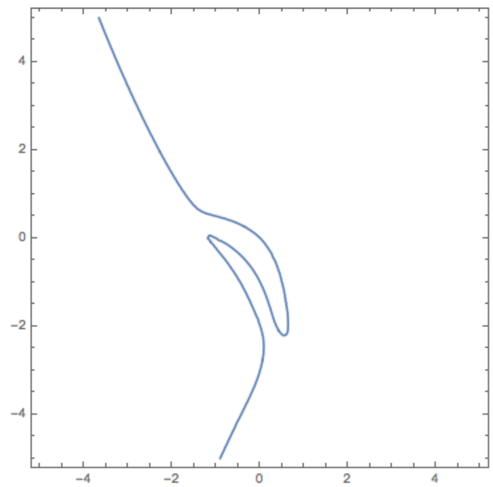


Figure 6. Real Tutte zeros: the Petersen graph

It also seems interesting to complexify x and y variables, and to consider the null variety of $T_G(x, y)$ in $\mathbb{C}^2$. In two dimensional complex space, representing the zero sets is more difficult. The following images show the zero sets for the real (blue) and imaginary parts (red) of the Tutte polynomial of various random regular graphs. The zero sets of the functions are the intersections of these curves.

In Figure 7 below, complex zeros of T_G were plotted, for the following random regular graphs: (A) a 3-regular graph with 8 vertices in the space with x complex and y real; (B) a 3-regular graph with 8 vertices in the space with y complex and x real; (C) a 3-regular graph with 10 vertices in the space with x complex and y real; (D) a 3-regular graph with 10 vertices in the space with y complex and x real.

In Figure 8 below, complex zeros of T_G were plotted, for the following random regular graphs: (A) a 4-regular graph with 8 vertices in the space with x complex and y real; (B) a 4-regular graph with 8 vertices in the space with y complex and x real; (C) a 4-regular graph with 9 vertices in the space with x complex and y real; (D) a 4-regular graph with 9 vertices in the space with y complex and x real.

In Figure 9 below, complex zeros of T_G were plotted, for the following random regular graphs: (A) a 4-regular graph with 10 vertices in the space with x complex and y real; (B) a 4-regular graph with 10 vertices in the space with y complex and x real; (C) a 5-regular graph with 8 vertices in the space with x complex and y real; (D) a 5-regular graph with 8 vertices in the space with y complex and x real.

6. Ribbon graphs: summary

Below we summarize our results about Bollobás-Riordan polynomials associated to ribbon graphs. In the paper [JMNT], the authors studied limiting distribution of coefficients of rank polynomials for random sparse graphs. In the current paper, we would like to extend those results to the case of oriented *ribbon graphs*, which are graphs with a cyclic orientation of edges at each vertex. Such graphs arise in the study of knot invariants, in quantum field theory and in other areas.

A natural extension of the rank polynomial to ribbon graphs is the *Bollobás-Riordan* polynomial ([BR01, BR02]), which is a polynomial in 3 variables. First, we

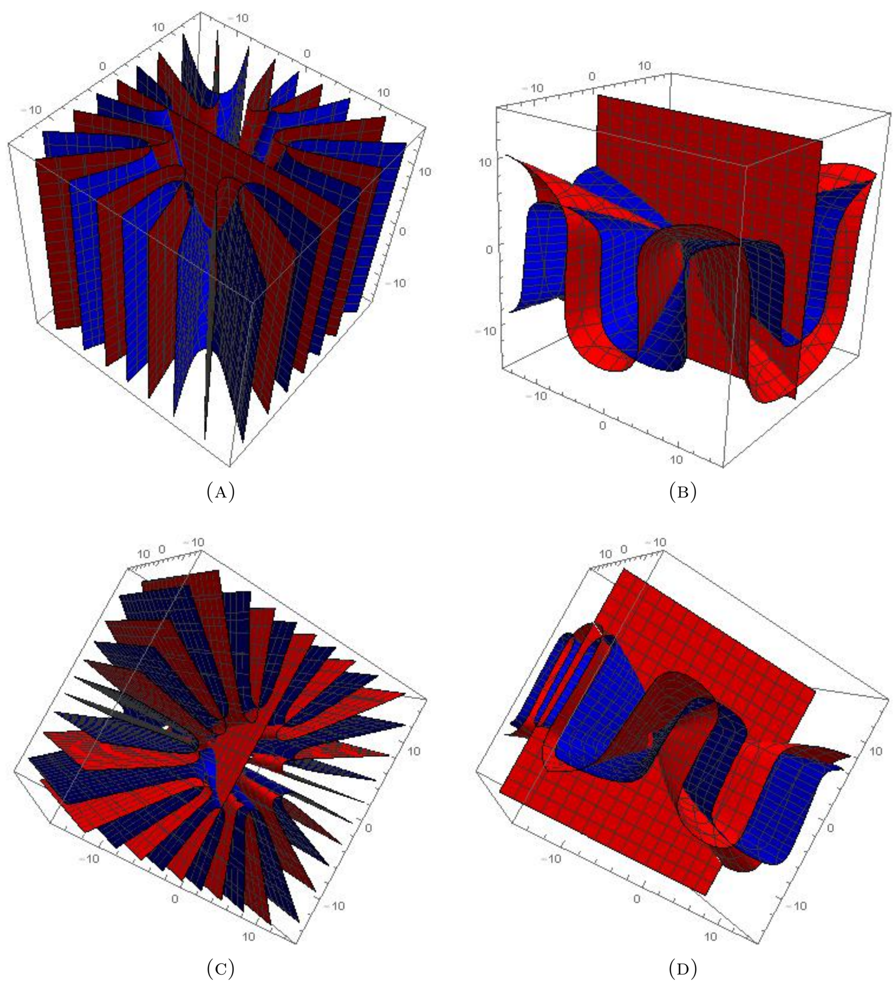


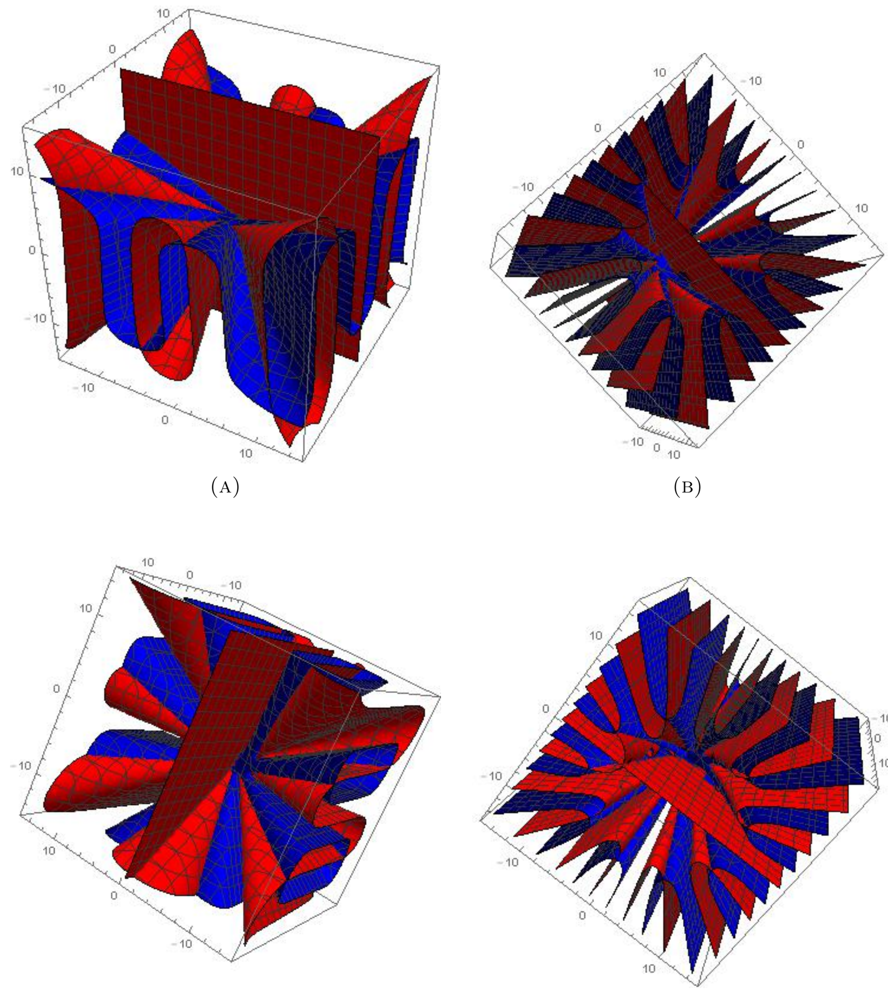
FIGURE 7. Complex zeros of the Tutte polynomial: 3-regular graphs

define the normalized coefficient measures for such polynomials. Next, we extend the definition of Benjamini-Schramm convergence from graphs to ribbon graphs. Finally, we extend some of the results in [JMNT] to the coefficients of Bollobás-Riordan polynomials for sequences of ribbon graphs that converge in the Benjamini-Schramm sense. Next, we compute the limit measure for sequences of random ribbon graphs arising from random edge orientations of regular graphs, see e.g. [Gam] and [Ch-Pit].

7. Ribbon graphs and "left-hand turn" surfaces

Below, we discuss ribbon graphs the associated left hand turn (LHT) surfaces.

An *orientable* ribbon graph is a graph embedded on an orientable surface such that every face such that every face of the resulting polyhedral surface is contractible. In this paper, we restrict ourselves to *orientable* ribbon graphs; this



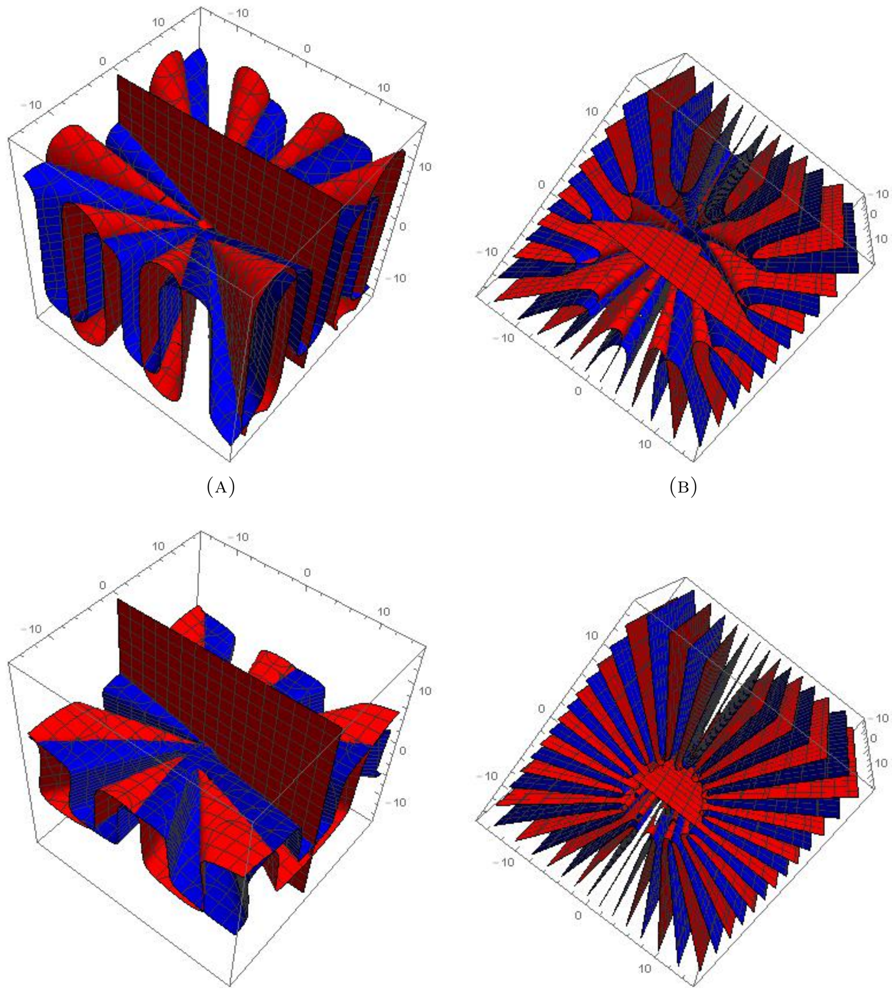
(A) (B) (C) (D)

FIGURE 8. Complex zeros of the Tutte polynomial: 4-regular graphs

simplifies our exposition. We refer to [Chm] and [MY] for more detailed description of ribbon graphs, including discussion about *non-orientable* ribbon graphs, sometimes called *Möbius* graphs.

Let G be a graph embedded on an orientable surface S. A choice of orientation on S defines a cyclic ordering of edges at every vertex; we denote such an ordering by O. Conversely, given a graph G with a cyclic ordering O of edges at every vertex, one can reconstruct in a canonical way an oriented surface S (considered in the papers of Brooks, Makover, Monastyrski [BM01, BM04, BrMon] and Gamburd [Gam]) as follows.

First, consider the *left-hand turn paths* defined by (G, O): start along an oriented edge, then "turn left" according to the cyclic orientation at the head of the oriented



(A) (B)

(C) (D)

FIGURE 9. Complex zeros of the Tutte polynomial: 4- and 5-regular graphs

edge, and continue until you get a closed cycle. The set of directed edges of G decomposes into a disjoint union of such cycles. Those cycles correspond to *boundary components* of a polyhedral surface $S(G, O)$, obtained by filling in every cycle with a disk. We call the corresponding surface the *left hand turn* (LHT) surface.[2]

The surface $S(G, O)$ can be defined equivalently as follows: cut every edge of G in the middle, and consider the corresponding "half-edges." The cyclic ordering of half-edges at every vertex determines a permutation β of the set of $2E(G)$ half-edges; a cycles in the cyclic decomposition of β consists of the half-edges incident to a given vertex, with the cyclic ordering determined by O. The cycle structure of

[2] Given a combinatorial graph G, there are clearly $\prod_{v \in V(G)}(deg(v)-1)!$ choices of the orientation O, and of the corresponding orientable ribbon graphs.

β coincides with the degree sequence of G. Another permutation α is an involution: it interchanges the half-edges that form a given edge. The cycle structure of α is $(2, 2, \dots, 2)$. One can show ([Chm, Gam, BM01, BM04, BrMon]) that the left-hand turn paths defined by (G, O) correspond to cycles in the cyclic decomposition of $\beta\alpha$. We discuss this further in Section 9.

We remark that for any subgraph F of G, a cyclic orientation $O(G)$ of edges of G incident to a given vertex u induces a cyclic orientation $O(F)$ of the edges of F incident to u. Accordingly, given an oriented ribbon graph (G, O), one can canonically define a ribbon subgraph $(F, O(F))$.

An orientation O is called *prime* if (G, O) has precisely *one* LHT path (equivalently, the surface $S(G, O)$ has one boundary component). The following result was shown in [BM01, Xu]; see also [BrMon]. Suppose that d is odd, and $n \equiv 2 (\bmod 4)$; or that n odd, and $d \equiv 2 (\bmod 4)$. Let G be a d-regular graph, and T a spanning tree of G such that $G \setminus T$ is connected (or more generally, so that each component of $G \setminus T$ has an even number of edges). Then G admits a prime orientation. Moreover, this last condition holds with asymptotic probability one as $n \to \infty$.

8. Bollobás-Riordan polynomials

We refer to the papers by [Ch-Pak] and [Mof] for the definition of Bollobás-Riordan polynomial. Let (G, O) be a ribbon graph, as described in Section 7. Given a subset F of edges of G, we define a subgraph of G as follows: we keep all the vertices of G, but only the edges from F; we shall call the corresponding subgraph F as well.

We denote by $|V(G)|$ the number of vertices of G; by $|E(G)|$ the number of its edges; and by $k(G)$ the number of its connected components. Also, we denote by $r(G) = |V(G)| - k(G)$ the rank of G; by $\mathrm{null}(G) = |E(G)| - r(G)$ the nullity (or co-rank) of G.

As discussed in Section 7, to every ribbon graph G we can canonically associate a surface with boundary $S(G, O)$ (whose boundary components correspond to the left hand turn paths defined by (G, O)). This coincides with the definition given in [Ch-Pak, §2]. We denote the number of its boundary components by $bc(G, O)$. We also remark that for any subgraph F of G, a cyclic orientation $O(G)$ of ribbon "half-edges" of G incident to a given vertex u induces a cyclic orientation $O(F)$ of ribbon "half-edges" of F incident to u. Accordingly, we can canonically define a ribbon graph $(F, O(F))$.

The *Bollobás-Riordan polynomial* is defined as follows. Denote by $\mathcal{F}(G)$ the set of all spanning subgraphs of G.

$$BR_{(G,O)}(x, y, z) = \sum_{F \in \mathcal{F}(G)} x^{r(G)-r(F)} y^{\mathrm{null}(F)} z^{k(F)+\mathrm{null}(F)-bc(F)}. \tag{8.1}$$

We remark that if we set $z = 1$, we get the following identity:

$$BR_{(G,O)}(x, y, 1) = x^{r(G)} R_G(1/x, y),$$

where $r(G) = |V(G)| - 1$ and R_G denotes the rank polynomial of G as defined in [JMNT]. We remark that $\deg(x) \le |V(G)| = n, \deg(y) \le |E(G)| = m$. We now make some elementary remarks about $\deg(z)$.

Let F be a subgraph of G with k connected components, say $C_1, \dots, C_k$. Let the component C_i have n_i vertices, m_i edges and b_i boundary components. The power of z in the term corresponding to F is given by $2k(F) + |E(F)| - |V(F)| - bc(F)$, where $2k(F) = 2k$ is twice the number of connected components of F; $|E(F)| = \sum_{i=1}^{k} m_i \leq m = |E(G)|$; $|V(F)| = |V(G)| = n$; and $bc(F) = \sum_{i=1}^{k} b_i \geq k$. We remark that $k \leq n$. Accordingly,

$$\deg(z) \leq 2k + m - n - k = m + k - n \leq m = |E(G)|. \tag{8.2}$$

For orientable ribbon graphs (G, O) considered in this paper, the number of boundary components $bc(F)$ of a subgraph F coincides with the number of LHT paths of F with an induced orientation (an isolated vertex of F is defined to correspond to one LHT path).

We note that the sum of all the coefficients of $BR_{(G,O)}(x, y, z) = 2^{|E(G)|} = 2^m$. Accordingly, if

$$BR_{(G,O)}(x, y, z) = \sum_{(i,j,k)} \rho(i, j, k) x^i y^j z^k,$$

we define the *normalized coefficient measure* of $BR(G, O)$ by

$$\mu_{BR}(G, O) = \frac{1}{2^m} \sum_{(i,j,k)} \rho(i, j, k) \delta \left(\frac{i}{n}, \frac{j}{m}, \frac{k}{m} \right). \tag{8.3}$$

By the previous remarks, $\mu_{BR}(G, O)$ is a probability measure supported in the unit cube $[0, 1]^3 \subset \mathbb{R}^3$.

We would like to study limit points of the measures $\mu_{BR}(G_\ell, O_\ell)$ for sequences (G_ℓ, O_ℓ) of sparse graphs with increasing number of vertices; we remark that by compactness, limit points will exist for arbitrary sequences of graphs, but we restrict ourselves to sparse graphs in this paper.

We finally remark that for an oriented ribbon subgraph F of an oriented ribbon graph G, the power of z in (8.1) is twice the genus $2\gamma(F, O)$ of the surface $S(F, O)$, see e.g. [Chm, p. 4]. It follows that the largest degree of z is attained by $F = G$.

$$\deg_z(BR_{(G,O)}(x, y, z)) = 2\gamma(G, O). \tag{8.4}$$

9. Random graphs with orientations

In this section, we apply some of the results obtained in [Gam] and [Ch-Pit] to the study of Bollobás-Riordan polynomials.

Random d-regular graphs with orientation considered in [Gam, §3,4] were described in Section 7 using permutations β and α. For such graphs, the cycle structure of β is $(d, d, \ldots, d)$. The cycle structure of α is $(2, 2, \ldots, 2)$. The LHT paths (corresponding to the faces of $S(G, O)$) correspond to the cycles of $\beta\alpha$. To state the next result, we introduce the following notation: A_N denotes the alternating subgroup (of the permutation group S_N); C_d denotes the conjugacy class in A_N consisting of permutations whose cycle decomposition is the product of (N/d) disjoint d-cycles; the convolution of probability measures on S_N is denoted by $*$. Finally, the total variation distance between probability measures μ, ν on $G = A_N$ is defined by $||\mu - \nu|| = \max_{A \subset G} |\mu(A) - \nu(A)|$.

The main result in [Gam] is the following theorem ([Gam, Thm. 4.1]):

Theorem 9.1. *Let $N = dn$ and let P_d denote the probability measure on A_N supported on C_d. Let U denote the uniform distribution on A_N. Then for $d \geq 3$,*

$$\lim_{n \to \infty} ||P_d * P_2 - U|| = 0.$$

This allowed to determine the asymptotic behaviour of the number of LHT paths for random cubic graphs G with a random orientation O at every vertex, cf. [Gam, Corollary 5.1].

Theorem 9.2. *Let $L(n)$ denote the number of LHT paths in a random cubic graph on n vertices with random orientation. Then, as $n \to \infty$, $\mathbb{E}(L(n)) = \log(3n) + \gamma + o(1)$ and $\mathrm{Var}(L(n)) = \log(3n) + \gamma - \pi^2/6 + o(1)$, where $\gamma = 0.5772\ldots$ is Euler's constant. Further, $(L(n) - \log n)/\sqrt{\log n}$ converges to standard normal distribution $N(0,1)$.*

The corresponding result for random regular graphs is [Gam, Corollary 4.1].

Theorem 9.3. *The distribution of LHT paths for random regular graphs with random orientation converges to Poisson-Dirichlet distribution.*

These results were extended in [Fl-P, Ch-Pit]. Below we formulate the results in [Ch-Pit] in a form that is convenient for applications to BS convergent sequences of graphs. We assume all vertices of our graphs will have degree d satisfying $3 \le \delta \le d \le D$ (minimum degree $\delta \ge 3$; maximum degree D). It follows from the definition of BS convergence that there exist *limiting proportions* $b_\delta, \ldots, b_D$ of vertices of degree $\delta, \ldots, D$ respectively.

9.1. **Random oriented graphs with a given degree sequence.** The results in [Ch-Pit] can be reformulated to provide a model for random graphs with orientations, generalizing the model considered in [Gam].

Let G_j have n_j vertices. Let $(n_{j,\delta}, \ldots, n_{j,D})$ be a sequence that is realizable as a degree sequence of a graph with n_j vertices; we have $\sum_{k=\delta}^{D} n_{j,k} = n_j$. We assume that

$$\lim_{j\to\infty} n_{j,k}/n_j = b_k, \qquad \delta \le k \le D.$$

Let H_j denote the half-edges of G_j; we have $N_j := |H_j| = \sum_{k=\delta}^{D} k n_{j,k} = 2E(G_j)$. We remark that

$$N_j \asymp 2n_j \sum_{k=\delta}^{D} k b_k \tag{9.1}$$

Consider now a conjugacy class β in the alternating group S_{N_j} whose cycle structure is

$$C_{\mathbf{m}} := (n_{j,\delta} C_\delta, \ldots, n_{j,D} C_D);$$

it determines the cyclic orientation O_j at the vertices.

Let α be a permutation whose cycle structure is $C_2 := (2, 2, \ldots, 2)$ (it determines the edges of G_j). We are interested in the cycle structure of the product $\beta\alpha$. Denote by $P(\mathbf{m}, j) := P(n_{j,\delta}, \ldots, n_{j,D})$ the probability measure supported on the conjugacy class of $C_{\mathbf{m}}$. Let $P(2, j)$ denote the measure P_2 on S_{N_j} supported on the conjugacy class $(2, \ldots, 2)$. An analogue of Theorem 9.1 in this setting was proved in [Ch-Pit, Theorem 2.2], who determined the asymptotic distribution of $P(\mathbf{m}, j) * P_2$. We state their result below.

Theorem 9.4. *Let $P(\mathbf{m}, j)$ and $P(2, j)$ be as above. Let U_j denote the uniform distribution on A_{N_j} if $C_{\mathbf{m}}$ and C_2 are of the same parity; and the uniform distribution on $S_{N_j} \setminus A_{N_j}$ if $C_{\mathbf{m}}$ and C_2 are of different parity. Then as $N_j \to \infty$, we have*

$$||P(\mathbf{m}, j) * P(2, j) - U_j|| = O(N_j^{-1}).$$

The following results follow as corollaries (see [Ch-Pit, §3]). We note that the construction in [Ch-Pit] is *dual* to the construction in the current paper: polygons glued along pairs of sides to form a surface in [Ch-Pit] correspond to cyclically oriented edges around a vertex in our paper. Accordingly, the number of vertices in [Ch-Pit, Thm. 3.1] is equal to the number of faces in our paper.

In the next statement, we keep the notation of Theorem 9.4. Below, $\mathbb{P}$ denotes the uniform measure on the set of LHT surfaces (G_j, O) considered in 9.4; recall that $\mathbb{P} = P(\mathbf{m}, j) * P(2, j)$. Also, let F_j denote the number of faces in (G_j, O); those faces are in bijection with cycles in the product of two permutations. Let C_N^e (resp. C_N^o) denote the total number of cycles of the permutation chosen uniformly at random from all even permutations (respectively all odd permutations) in S_N. The next result is a restatement of [Ch-Pit, Thm. 3.1].

Proposition 9.5. *If $C_{\mathbf{m}}$ and C_2 are of the same parity, then $||\mathbb{P}(F_j - C^e_{N_j})|| = O(N_j^{-1})$; if $C_{\mathbf{m}}$ and C_2 are of the opposite parity, then $||\mathbb{P}(F_j - C^o_{N_j})|| = O(N_j^{-1})$.*

Denote by C_N the number of cycles of a random permutation in S_N. It is known that

$$\mathbb{E}[C_N] = \sum_{j=1}^{N} 1/j = \log N + O(1); \qquad \mathrm{Var}[C_N] = \sum_{j=1}^{N} (1/j)(1 - 1/j) = \log N + O(1).$$

We continue the summary of results from [Ch-Pit, §3,4]. The first result concerns the number X_j of components of the surface $S(G_j, O_j)$, [Ch-Pit, Thm. 4.1]:

$$\mathbb{P}(X_j = 1) = 1 - O(N_j^{-1}).$$

Fix $a \in \mathbb{N}$. The genus γ_j of $S(G_j, O_j)$ was determined in [Ch-Pit, Thm. 4.2].

Theorem 9.6. *With notation as above, for all admissible ℓ, the genus $\gamma_j = \gamma$ of $S(G_j, O_j)$ satisfies (here we denote $N_j = N, m_j = m$)*

$$\mathbb{P}(\gamma_j = 1 + N/4 - m/2 - \ell/2) = \frac{(2 + O(\log^{-1/2} N))}{\sqrt{2\pi \mathrm{Var}[C_N]}} \exp\left(-\frac{(\ell - \mathbb{E}[C_N])^2}{2\mathrm{Var}[C_N]}\right),$$

where ℓ is admissible provided $(\ell - \mathbb{E}[C_N])/\sqrt{\mathrm{Var}[C_N]} \in [-a, a]$.

Using (9.1), find that

$$\mathbb{E}[\gamma(G_j, O_j)] \asymp 1 + \frac{n_j}{2}\left(\sum_{k=\delta}^{D} k b_k - 1\right) - \log\left(2 n_j (\sum_{k=\delta}^{D} k b_k)\right) \tag{9.2}$$

By (8.4), we find that

$$\mathbb{E}[\deg_z(BR_{G_j}(x, y, z))] = 2\mathbb{E}[\gamma(G_j, O_j)]$$

10. Convergence of the coefficient measures of Bollobás-Riordan polynomials

In this section, we extend the results of [JMNT] showing that the normalized coefficient measures of rank polynomials converge to a delta function, provided a sequence of the corresponding (bounded degree) graphs converges in the sense of Benjamini-Schramm convergent sequences. We extend that result to the normalized coefficient measures (8.3), provided a sequence of (bounded degree) oriented ribbon graphs converges as in (10.1) defined below.

10.1. **BS Convergence for oriented ribbon graphs and LHT surfaces.** In the section 9, we considered random oriented graphs with a given degree sequence. In this section, we would like to study more general sequences of oriented graphs. We concentrate on two cases:

(a) Take a sequence of graphs G_j (of minimal degree 3) that converges BS, and put a random cyclic orientation of the edges at every vertex.
(b) Generalize the notion of BS convergence to oriented graphs.

Motivated by our approach in [JMNT], we extend the definition of Benjamini-Schramm convergence "left-hand turn" surfaces $S(G, O)$; this coincides with the BS convergence for oriented ribbon graphs.

In what follows, we will consider an extension of Benjamini-Schramm convergence to ribbon graphs and oriented graphs. Everything remains the same, except that every instance of rooted graph is replaced by rooted oriented or rooted ribbon graph. Isomorphisms of such graphs are required to also preserve the additional structure.

Let $\{(G_j, O_j)\}$ be a sequence of ribbon graphs; we restrict ourselves to graphs of bounded degree, and assume that $|V(G_j)| \to \infty$ as $j \to \infty$. Denote by $S(G_j, O_j)$ the corresponding LHT surfaces. By analogy with the usual Benjamini-Schramm convergence, we say that (G_j, O_j) converges to an infinite ribbon graph (G_∞, O_∞) with a given probability measure on its vertices, provided the following holds. Given $R \in \mathbb{N}$ and a finite ribbon graph α, denote by $\mathbb{P}_{(G_j,O_j)}(\alpha, R)$ the probability that a ball of radius R centered at a random vertex $u \in (G_j, O_j)$ is isomorphic to α.

Definition 10.1. The sequence (G_j, O_j) converges to (G_∞, O_∞) provided that for any $R \in \mathbb{N}$, and for any α, there exists $0 \leq P_\infty(\alpha, R) \leq 1$ such that $\mathbb{P}_{(G_j,O_j)}(\alpha, R) \to P_\infty(\alpha, R)$ as $j \to \infty$.

This is not the original definition given by Benjamini and Schramm, but it is equivalent in the case of graphs of uniformly bounded degree.

For what follows, instead or choosing the root from the uniform distribution, we choose the root from the stationary distribution, essentially choosing a uniformly random root edge. For sequences of graphs of uniformly bounded degree (and no isolated points), Benjamini-Schramm convergence according to the uniform distribution is equivalent to that under the stationary distribution. Indeed the probability measures in the two cases differ only by multiplication by a uniformly bounded, continuous function. Since the graph is oriented, we will more specifically choose a specific direction of traversal of the root edge. In the case of ribbon graphs, this will mean the side of the ribbon to the left as the edge is traversed in that direction. Let $\vec{uv}$ denote the edge uv traversed from u to v. Benjamini-Schramm convergence is stated in terms of functions of graphs rooted at vertices, however for this proof it is more convenient to consider graphs rooted at edges. The two are equivalent for locally finite graphs since a function f on the directed edges can be considered a function of the vertices by letting $f(v)$ be the average of the $f(\vec{vu})$, for arcs pointing out of v.

Let $L_{\vec{uv}}$ denote the length of the boundary component containing $\vec{uv}$. For an edge subgraph A, let $L^A_{\vec{uv}}$ denote the length of the boundary component in A containing $\vec{uv}$.

Theorem 10.2. *The sequence of coefficient measures of the Bollobás-Riordan polynomials of a Benjamini-Schramm convergent sequence of oriented ribbon graphs with uniformly bounded degree converges to a δ function.*

Proof. Consider a sequence (G_j, O_j) of oriented ribbon graphs of bounded degree, that converges in the sense of Definition 10.1; we denote the number of vertices of G_j by n_j, and the number of edges of G_j by m_j.

For each j, let A_j be a subset of the edge set of G_j chosen by including each edge uniformly and independently at random with probability $\frac{1}{2}$.

It is shown in Theorem 18 of [JMNT] that

- The measure associated to $|A_j|/n_j$ converges weakly to a δ function.
- The measure associated to $k(A_j)/n_j$ (where $k(A_j)$ is the number of connected components of A_j) converges weakly to a δ function.
- The density m_j/n_j converges to a non-zero real number.

This implies that the measures associated to $r(A_j)/m_j$ and $s(A_j)/m_j$ where r is the rank and s the co-rank of A_j converge weakly to δ functions. In Lemma 17 of [JMNT], it is shown that the subgraphs A_j themselves form a Benjamin-Schramm convergent sequence of un-oriented graphs. This result extends naturally to ribbon graphs. Similarly to Lemma 17, one could also formulate the convergence result in the language of probability measures on random subgraphs of (G_j, O_j); we leave it as an exercise.

It suffices to show that $\mathbb{E}\left[bc(A_j)/m_j\right]$ converges and $\operatorname{Var}\left(bc(A_j)/m_j\right) \to 0$. The expectations and variance here are taken with respect to the choice of a uniformly random edge subgraph A_j by independently removing each edge with probability $\frac{1}{2}$. The proof will be very similar to the treatment of the number of connected components in [JMNT]. Let us first note that

$$bc(A_j) = \sum_{(u,v):uv\in A_j} \left(L_{\vec{uv}}^{A_j}\right)^{-1}$$

and let

$$\ell^{A_j}(\vec{uv}) = \begin{cases} \left(L_{\vec{uv}}^{A_j}\right)^{-1}, & \text{if } uv \in A_j \\ 0, & \text{otherwise} \end{cases}.$$

Then $bc(A_j) = \sum_{(u,v):uv\in E(G_j)} \ell^{A_j}(\vec{uv})$. Define further

$$\ell_R^{A_j}(\vec{uv}) = \begin{cases} \ell^{A_j}(\vec{uv}), & \text{if } \ell^{A_j}(\vec{uv}) > \frac{1}{R} \\ 0, & \text{otherwise} \end{cases}.$$

The quantity $\ell_R^{A_j}(\vec{uv})$ depends only on what happens within the ball of radius R about uv and is within $1/R$ of $\ell^{A_j}(\vec{uv})$.

To show convergence of the expectation, we use linearity of expectation.

$$\begin{aligned} \mathbb{E}\left[\frac{bc(A_j)}{m_j}\right] &= \mathbb{E}\left[\sum_{\vec{uv}:uv\in E} \frac{1}{m_j}\ell^{A_j}(\vec{uv})\right] \\ &= \frac{1}{m_j} \sum_{\vec{uv}:uv\in E(G_j)} \mathbb{E}\left[\ell^{A_j}(\vec{uv})\right] \end{aligned}$$

We may now approximate $\mathbb{E}\left[\ell^{A_j}(\vec{uv})\right]$ using $\ell_R^{A_j}(\vec{uv})$. $\mathbb{E}\left[\ell^{A_j}(\vec{uv})\right]$ and $\mathbb{E}\left[\ell_R^{A_j}(\vec{uv})\right]$ differ by at most $\frac{1}{R}$. However, $\ell_R^{A_j}(\vec{uv})$ is entirely determined by the behaviour of A_j in the ball of radius R about uv. Hence $\mathbb{E}\left[\ell_R^{A_j}(\vec{uv})\right]$ is entirely determined by the ball of radius R about uv. Since the G_j have uniformly bounded degree, such a ball can possibly be isomorphic to only a finite number of graphs. Since the G_j are Benjamini-Schramm convergent, the probability that the ball of radius R about a randomly selected $\vec{uv}$ is isomorphic to any given graph converges. Thus $D_{R,j} = (1/m_j)\sum_{\vec{uv}} \mathbb{E}\left[\ell_R^{A_j}(\vec{uv})\right]$ converges to a limit D_R as $j \to \infty$. The sequence of D_R (indexed by R) is Cauchy, and thus converges to a limit D as $R \to \infty$. Let $\epsilon > 0$ be given and choose $R > 3/\epsilon$. Then, for all j, $\frac{1}{m_j}\sum_{\vec{uv}} \mathbb{E}\left[\ell^{A_j}(\vec{uv})\right]$ is within $\epsilon/3$ of $D_{R,j}$. Choose J large enough that for all $j \geq J$, $D_{R,j}$ is within $\epsilon/3$ of D_R, which is within $1/R$ of D. Then, summing up all the errors for all $j \geq J$, $(1/m_j)\sum_{\vec{uv}} \mathbb{E}\left[\ell^{A_j}(\vec{uv})\right]$ is within ϵ of D. Thus $(1/m_j)\sum_{\vec{uv}} \mathbb{E}\left[\ell^{A_j}(\vec{uv})\right]$ converges to D.

For the variance, let $\epsilon > 0$ be given. Let $R > \frac{16}{\epsilon}$ and consider again $\ell_R^{A_j}(\vec{uv})$. Since the graph has bounded degree, there is a bound on the number of edges at distance at most R from any point. Hence, for all but $O(m_j)$ pairs (uv, wx), $\ell_R^{A_j}(\vec{uv})$ and $\ell_R^{A_j}(\vec{wx})$ are independent. Hence $\mathrm{Var}\left(\sum_{(u,v):uv\in E(G_j)} \ell_R^{A_j}(\vec{uv})\right) = O(m_j)$. Now let $f_R^{A_j}(\vec{uv}) = \ell^{A_j}(\vec{uv}) - \ell_R^{A_j}(\vec{uv})$ and note that this is always bounded above by $1/R < \epsilon/16$. We note that

$$bc(A_j) = \sum_{(u,v):uv\in E(G_j)} \ell_R^{A_j}(\vec{uv}) + \sum_{(u,v):uv\in E(G_j)} f_R^{A_j}(\vec{uv})$$

and so

$$\begin{aligned}
\mathrm{Var}\left(bc(A_j)\right) &= \mathrm{Var}\left(\sum_{(u,v),uv\in E(G_j)} \ell_R^{A_j}(\vec{uv}) + \sum_{(u,v),uv\in E(G_j)} f_R^{A_j}(\vec{uv})\right)\\
&= Var\left(\sum_{(u,v):uv\in E(G_j)} \ell_R^{A_j}(\vec{uv})\right) + \mathrm{Var}\left(\sum_{(u,v):uv\in E(G_j)} f_R^{A_j}(\vec{uv})\right) +\\
&\quad Cov\left(\sum_{(u,v):uv\in E(G_j)} f_R^{A_j}(\vec{uv}), \sum_{(u,v):uv\in E(G_j)} \ell_R^{A_j}(\vec{uv})\right)
\end{aligned}$$

We know that the first term is $O(m_j)$. The other two are each less than $(\epsilon/8)4m_j^2$ and so their sum is less than ϵm_j^2. Choose a graph sufficiently far in the sequence G_j that for all subsequent graphs, the first term is smaller than $(\epsilon/2)m_j^2$. Hence $\mathrm{Var}\left(bc(A_j)/m_j\right) < 3\epsilon/2$ past a certain point in the sequence. This holds for all $\epsilon > 0$ and so $\mathrm{Var}\left(bc(A_j)/m_j\right) \to 0$. This finishes the proof. □

This theorem deals with the case of deterministic Benjamini-Schramm convergent sequences of oriented ribbon graphs. This corresponds to point (b) stated at the start of this section. There can also be Benjamini-Schramm convergent sequences of random ribbon graphs. In these, each $\mathcal{G}_j$ is a distribution on the graphs of a given size. The random vertex is selected by first choosing a graph G_j from $\mathcal{G}_j$ and then selecting a random vertex from it. Point (a) given at the start of the section is such a case. There is an underlying Benjamini-Schramm convergent sequence of graphs G_j. It is made into an oriented ribbon graph by assigning independently and uniformly at random a cyclic ordering to the edges at every vertex. This gives us a sequence $(G_j, \mathcal{O}_j)$ of random ribbon graphs. They are Benjamini-Schramm convergent since, for all oriented ribbon graphs α with underlying graph α' and radii R, $\mathbb{P}_{(G_j,\mathcal{O}_j)}(\alpha, R) = \mathbb{P}_{G_j}(\alpha', R)\mathbb{P}(\alpha|\alpha')$ where $\mathbb{P}(\alpha|\alpha')$ denotes the probability of obtaining α from α' by assigning a cyclic ordering of edges to every vertex. The first factor converges as $j \to \infty$ since the G_j were Benjamini-Schramm convergent. The second factor does not depend on j.

The theorem and proof as stated do not apply to Benjamini-Schramm convergent sequences of random oriented ribbon graphs. However, the proof can be modified to account for the r andomness that arises from giving a cyclic orientation O_j to the edges around each vertex in a Benjamini-Schramm convergent sequence of random graphs. Replace each instance of $L^{A_j}_{\vec{uv}}$, $\ell^{A_j}(\vec{uv})$ and $\ell^{A_j}_R(\vec{uv})$ by $L^{A_j,O_j}_{\vec{uv}}$, $\ell^{A_j,O_j}(\vec{uv})$ and $\ell^{A_j,O_j}_R(\vec{uv})$ respectively to represent the fact that the length of boundary components depends also on the cyclic ordering which is now also random. The crucial properties that $\ell^{A_j,O_j}(\vec{uv})$ and $\ell^{A_j,O_j}_R(\vec{uv})$ are within $1/R$ of each other and that $\ell^{A_j,O_j}_R(\vec{uv})$ depends only on the behaviour of A_j and O_j in the ball of radius R about uv remain unchanged. Take the expectation and variance over both A_j and O_j. The proof proceeds exactly as before. Thus Theorem 10.2 applies to the sequences of ribbon graphs described in point (a) at the start of this section.

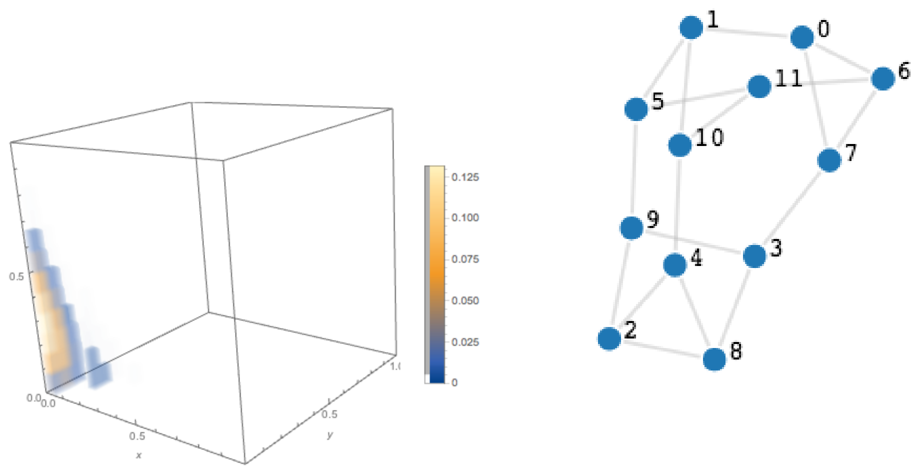
In Section 6 of [JMNT], the precise coordinates of the limiting δ function of the coefficient measure of the rank polynomial were found in the case of random regular graphs. We now find the z-axis coordinate for the limit of the coefficient measure of the Bollobás-Riordan polynomial for oriented ribbon graphs obtained from random regular graphs by adding a cyclic ordering of the edges at every vertex. Random regular graphs are not a sequence of Benjamini-Schramm convergent deterministic graphs. However, as discussed in section 6 of [JMNT], they converge to a fixed graph and thus are almost uniformly Benjamini-Schramm convergent as defined on page 17 of the same paper, allowing them to be treated similarly to a deterministic sequence. By 8.4, the degree $\deg_z(BR_{(G,O)}(x, y, z)) = 2\gamma(G, O)$, where $\gamma(G, O)$ denotes the genus of the orientable LHT surface $S(G, O)$. By Euler's formula, it is equal to $2+|E(G)|-|V(G)|-L(G, O)$, where $L(G, O)$ denotes the number of LHT paths of (G, O); it is also equal to the number of faces of $S(G, O)$. By Proposition 9.5, for d-regular graphs on n vertices with random orientation, $L(G, O)$ grows logarithmically in n. We have $|V(G) = n, |E(G)| = dn/2$. The normalized z-coordinate of the δ function is equal to $\mathbb{E}[2\gamma(G, O)]/|E(G)|$. To leading order in n, it is asymptotic to $[n(d/2 - 1)]/[dn/2]$. After cancellations, we find that the normalized z-coordinate of the limiting δ function is equal to $(d - 2)/d$.

11. Numerical investigations: coefficients and zeros of Bollobás-Riordan polynomials

To compute Bollobás-Riordan on random graphs, we generate random cyclic orientations of half-edges around each vertex, and then use the SAGE ribbon graph library to count the number of boundary components for each induced subgraph we must sum over. The SAGE code is included in the section 12. Since the evaluation of the Tutte polynomial can yield the number of 3-colourings, which is a $\#P$-complete problem, the computation of the Tutte polynomial is $\#P$-hard (the class $\#P$ is a complexity class that contains function problems of counting solutions that correspond to underlying NP decision problems). Therefore, computing the Tutte and Bollobás-Riordan polynomials is not feasible for large graphs in reasonable amounts of time. Note that the computation employed below is exponential in the number of edges, so the problem quickly becomes intractable as the the graphs grow. However, the computation is highly parallelizable (we can sum over induced graphs separately on different processors and then sum up the polynomials at the end), which can yield notable increases in speed and potentially allow us to study slightly larger graphs.

We first provide pictures of the coefficient measures for Bollobás-Riordan polynomials.

Figure 10 shows the coefficient measure for a ribbon graph obtained by choosing a random orientation on a 3-regular graph on 12 vertices; the graph itself is shown as well.

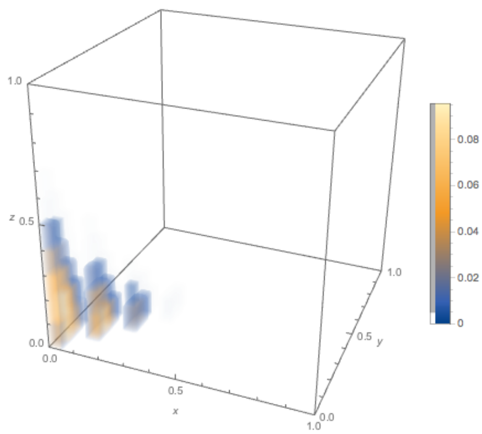


(A) (B)

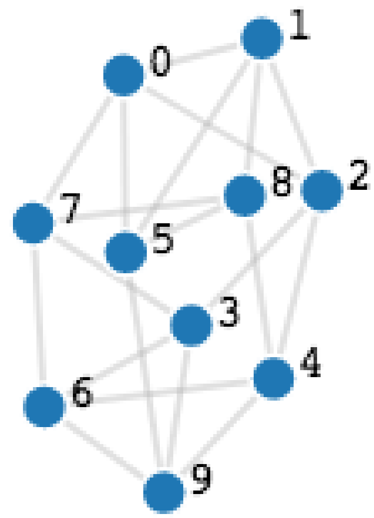
FIGURE 10. BR Coefficients: 3-regular graph on 12 vertices

Next, Figure 11 shows the coefficient measure for a ribbon graph obtained by choosing a random orientation on a 4-regular graph on 10 vertices; the graph itself is shown as well.

Next, we include the pictures of the zero sets in $\mathbb{R}^3$ of Bollobás-Riordan polynomials. Figure 12 shows the zero set for a 3-regular graph on 12 vertices; it is the same graph as in Figure 10.



(A)



(B)

FIGURE 11. BR Coefficients: 4-regular graph on 10 vertices

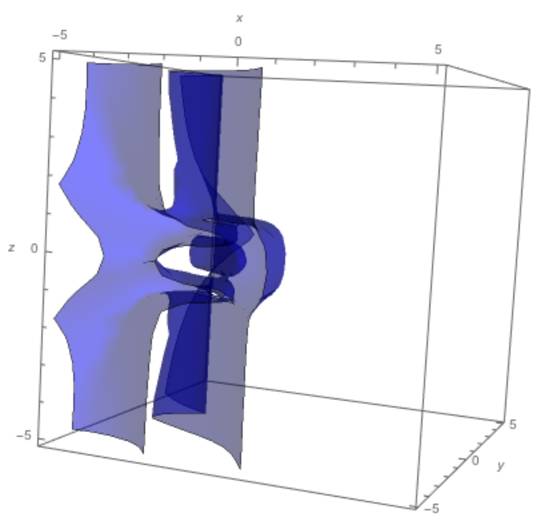


FIGURE 12. Bollobás-Riordan zeros: 3-regular graph on 12 vertices

Figure 13 shows the zero set for a 4-regular graph on 10 vertices; it is the same graph as in Figure 11.

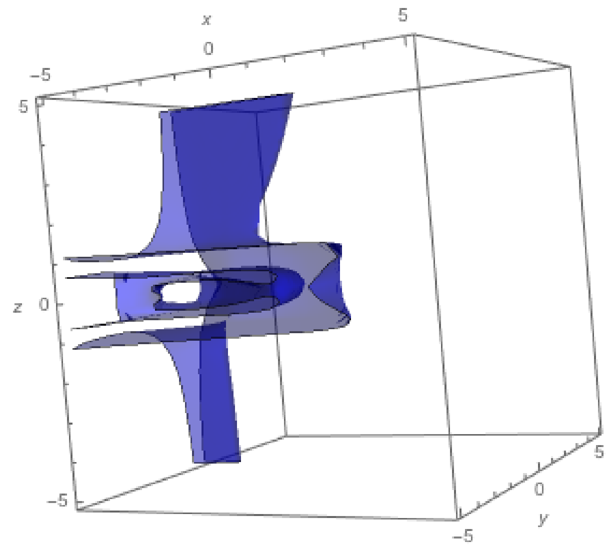


Figure 13. Bollobás-Riordan zeros: 4-regular graph on 10 vertices

12. Conclusion

In conclusion, we would like to discuss some natural questions that arise from the results in [JMNT] and the present paper.

Problem 1. It seems natural to generalize results in [JMNT] and the present paper to higher-dimensional simplicial complexes; a natural generalization of rank and Bollobás-Riordan polynomials are the Kruskal-Renardi polynomials, see e.g. [KR], [BBC]. We expect that an analogue of Theorem 10.2 holds for (suitably defined) sequences of converging CW complexes, at least under suitable bounded degree restrictions.

Problem 2. We hope to establish large deviation results for the coefficient measures of rank polynomials (for random regular graphs, and possibly for more general models of random graphs with degrees). This is work in progress with O. Angel, C. MacRury and L. Silberman. We also hope to establish similar results for the coefficient measures Bollobás-Riordan polynomials (e.g. for random regular graphs with random orientations, and possibly for more general models considered e.g. in [Ch-Pit]).

Problem 3. It seems natural to study asymptotic behaviour of the zero sets of rank, BR and related polynomials; in particular beyond the asymptotic behaviour "at infinity" discussed briefly in Section 5. Since the degree of the polynomials is growing, their nodal sets should be rescaled appropriately. It could be interesting to compare their behaviour with the asymptotic behaviour of nodal sets of high energy Laplace eigenfunctions, cf. e.g. [Zel] for a recent survey.

Problem 4. It seems very interesting to explore in more detail possible applications to Statistical Physics, e.g. in connection to the q-state Potts model on graphs.

Problem 5. It seems interesting to explore possible connections to knot polynomials and asymptotic properties of knot invariants.

Problem 6. A natural question is to study the asymptotic behaviour of the coefficient measures and zero sets for sequences of *dense* graphs, removing the bounded degree restriction of [JMNT] and the present paper; one could consider various notions of convergence for such graphs (e.g. graphon convergence etc).

Appendix: Computer code

In this section we include the computer code used in the numerical investigations described in Section 11. The Mathematica commands we used to generate random regular graphs and their Tutte polynomials were described in section 4 and 5 .

For Bollobás-Riordan polynomials, we used SAGE to generate random graphs with random orientations, and standard Mathematica commands to plot the coefficient densities and zeros of the corresponding polynomials. We include the SAGE code below.

```
from copy import deepcopy

G = graphs.RandomRegular(3,20) #uniform random 3-regular graph on 20
vertices
V = G.vertices()
dartlist = [[] for _ in range(len(V))]
#We label each edge with a positive number k, and each half-edge
with either 2k or 2k-1. This allows us to pass between edges and
half-edges to make use of the ribbon graph library while we are
cycling over spanning edge subsets during the computation of the
Bollobas-Riordan polynomial.
r = [] #involution of darts (sends each dart to its other half)
for i, e in enumerate(G.edges()):
    dartlist[e[0]].append(2*i+1)
    dartlist[e[1]].append(2*i+2)
   G.set_edge_label(e[0], e[1], i+1)
    r.append([2*i+1,2*i+2])
E = G.edges()}
rank_G = len(V) - G.connected_components_number()
nullity_G = len(E) - rank_G
s = [] #cyclic permutations of darts in mutable list form,
generated uniformly at random
for dl in dartlist:
    shuffle(dl)
   s.append(dl)
edge_subsets = []
#generates a list of edge subsets
for i in range(1,1 << len(E)):
    subset = [E[bit] for bit in range(len(E))
    if i & (1 << bit) > 0]
    edge_subsets.append(subset)
#we now loop over all spanning subgraphs
x,y,z = var('x,y,z')
brpoly(x,y,z) = x^(rank_G);
for E_F in edge_subsets:
    F = Graph([V,E_F])
```

```
    removededges = [l[2] for l in
    list(Set(E).difference(Set(E_F)))]
    removeddarts = []
   r_F = []
    for e in removededges:
        removeddarts.append(2*e-1)
        removeddarts.append(2*e)
    for d in r:
        if d[0] not in removeddarts and d[1] not in removed darts:
            r_F.append(d);
    s_F = deepcopy(s)
    for i, dl in enumerate(s):
        for d in dl:
            if d in removeddarts:
                s_F[i].remove(d)
    s_F = [a for a in s_F if (a != [] and len(a) != 1)]
    sigma_F = PermutationGroupElement(''.join([str(tuple(a))
    for a in s_F]))
    rho_F = PermutationGroupElement(''.join([str(tuple(a))
    for a in r_F]))
    isolated = F.degree_histogram()[0]
    bc = RibbonGraph(sigma_F,rho_F).number_boundaries()
    bc += isolated
   components_F = F.connected_components_number()
    rank_F = len(V) - components_F
    nullity_F = len(E_F) - rank_F
brpoly = brpoly + x^(rank_G - rank_F) * y^(nullity_F) *
z^(components_F - bc + nullity_F)
 print(brpoly)
```

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