Lecture notes from a seminar of Professor Dmitry Jakobson. Typography (and all mistakes!) are due to Michael McBreen.

1 Elliptic Operators Associated to Generic Metrics

1.1 Introduction

A real symmetric matrix has simple (i.e. distinct) eigenvalues λ_i iff its discriminant $\prod_{i<j}(\lambda_i - \lambda_j)^2$ is nonzero. The discriminant happens to be sum of squares in the matrix elements (see [9]). For 2×2 matrices, for instance,

$$\text{Discriminant} \begin{pmatrix} a & c \\ c & b \end{pmatrix} = (a+b)^2 - 4(ab - c^2) = (a-b)^2 + 4c^2.$$

There are hence two constraints on the elements of a non-simple symmetric 2×2 matrix, so the non-simple matrices have codimension 2 in the space of symmetric matrices. The codimension of the set of $n \times n$ symmetric matrices with given eigenvalue multiplicity is known: see [20].

Self-adjoint elliptic operators[1] are analogous to positive-definite symmetric matrices, so it's natural to ask whether within a given family of self-adjoint elliptic operators, the non-simple ones have codimension two. The answer is often positive: see Teytel [17] and Arnold [2].

For instance, we can study the question

$$(\Delta - \lambda)\phi = 0$$

on $\Omega \subset \mathbb{R}^n$ or a riemannian manifold M, with Dirichlet boundary conditions. We then have Weyl's law:

$$\#\{\lambda_i < r\} \sim r^{\frac{n}{2}} \times \text{volume}(\Omega).$$

We can view Δ as an element of the following operator families:

- $\Delta + tV$ where V is a smooth potential function.
- Δ_Ω where we vary Ω by deforming its boundary.
- Δ_g where we vary the metric g with respect to which Δ is defined.[2]

[1] For the definitions and a few properties, see section 1.3.

[2] For instance, if we define a family of metrics on the torus by $g = e^{th(x,y)}(dx^2 + dy^2)$, then $\Delta_g = e^{-th(x,y)}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.

Albert [1] showed that eigenvalues of Δ are simple for a generic $\Delta + tV$, by breaking up higher dimensional eigenspaces with small perturbations of the operator. This is the subject of the next section.

1.2 Perturbing the spectrum

We show how to perturb the operator $\Delta + tV$ to obtain a simple spectrum, following Albert in [1]. Consider $\Delta + tV, \quad 0 \le t \le 1, \quad V \in C^\infty(M)$, λ an eigenvalue of Δ, E_λ the (finite) eigenspace for λ, $\dim(E_\lambda) = m_\lambda = m$.

As we vary t, each eigenvalue λ_t of $\Delta + tV$ varies continuously. In fact,

$$|\lambda(t+h) - \lambda(t)| \le h|V|.$$

This can be proved by the minimax principle. It follows that if we fix N, for small enough perturbations of t no new multiplicities appear in the region $\lambda < N$.

By a theorem of Rellich (see [13]) one can choose an orthonormal basis of E_λ depending on V, $\{\phi_{1,0}, \phi_{2,0}, \dots, \phi_{m,0}\}$, such that for all k we have

$$(\Delta+tV)(\phi_{k,0}+t\phi_{k,1}+t^2\phi_{k,2}+\dots) = (\lambda+t\lambda_{k,1}+t^2\lambda_{k,2}+\dots)(\phi_{k,0}+t\phi_{k,1}+t^2\phi_{k,2}+\dots). \tag{1}$$

where the $\phi_k(t) = (\phi_{k,0} + t\phi_{k,1} + t^2\phi_{k,2} + \dots)$ are orthonormal. Comparing the coefficients of t in (1), we get

$$V\phi_{k,0} + \Delta\phi_{k,1} - \lambda_{k,1}\phi_{k,0} - \lambda\phi_{k,1} = 0 \tag{2}$$

Contract with $\phi_{k,0}$ to get

$$\begin{aligned} 0 &= <\phi_{k,0}, V\phi_{k,0}> + <\phi_{k,0}, \Delta\phi_{k,1}> - <\phi_{k,0}, \lambda_{k,1}\phi_{k,0}> - <\phi_{k,0}, \lambda\phi_{k,1}> \\ &= <\phi_{k,0}, V\phi_{k,0}> + <\Delta\phi_{k,0}, \phi_{k,1}> -\lambda_{k,1} \\ &= <\phi_{k,0}, V\phi_{k,0}> -\lambda_{k,1}. \end{aligned}$$

The second equality follows from $||\phi_{k,0}||_2 = 1$ and $<\phi_{k,0}, \phi_{k,1}>= 0$, which in turn follows from $||\phi(t)||_2 = 1$ Contract (2) with $\phi_{k',0}$ $(k' \ne k)$ to get

$$\begin{aligned} 0 &= <\phi_{k',0}, V\phi_{k,0}> + <\phi_{k',0}, \Delta\phi_{k,1}> -\lambda_{k,1} <\phi_{k',0}, \phi_{k,0}> -\lambda <\phi_{k',0}, \phi_{k,1}> \\ &= <\phi_{k',0}, V\phi_{k,0}> +\lambda <\phi_{k',0}, \phi_{k,1}> -\lambda <\phi_{k',0}, \phi_{k,1}> \\ &= <\phi_{k',0}, V\phi_{k,0}> . \end{aligned}$$

Hence $\lambda_{k,1}$ are the eigenvalues of the operator $B = \Pi_\lambda m_V : E_\lambda \to E_\lambda$ where Π_λ is projection onto E_λ and m_V represents multiplication by V. If we find a V such that $\lambda_{k',1} \neq \lambda_{k,1}$ for some k', k, then for small t, $\lambda_{k'}(t) \neq \lambda_k(t)$. Moreover, for small enough perturbations, no new multiplicities can appear for $\lambda < N$ (for some arbitrarily large N). It follows that for such a V, a small perturbation will decrease the multiplicity of λ by at least 1. A sequence of such perturbations will then kill all the multiplicity for $\lambda < N$.

Hence we need only find an appropriate V. For instance, one can check that either $V = \phi_{1,0}\phi_{2,0}$ or $V = \phi_{1,0}^2 - \phi_{2,0}^2$ works. If we can show that such V are residual in C^∞, then the intersection of the sets of admissible V corresponding to the different λ will be dense by the Baire Category theorem (since the spectrum is countable) and hence we can choose a V which breaks up all E_λ simultaneously.

1.3 Elliptic Regularity (Phil Sosoe)

We discuss the regularity of elliptic operators (this section is self-contained).

Let $\alpha \in \mathbb{Z}^n$ have non-negative entries. We define the operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

For $\zeta \in \mathbb{R}^n$, we define

$$< \zeta >= (1 + |\zeta|^2)^{\frac{1}{2}}.$$

This behaves like the norm function for large vectors, but never vanishes.

We call

$$S = \{f \in C^\infty(\mathbb{R}^n) \ \mid \ \forall \ \alpha, \ \forall \ p(x) \in \mathbb{R}[x_1, \dots, x_n], \ \ \sup(p(x) D^\alpha f(x)) < \infty\}$$

the Schwarz space, and give it the obvious Frechet seminorms. Schwarz functions (and their derivatives) decay faster than any inverse polynomial at infinity.

One can show that the Fourier transform F is an isomorphism of the Schwarz space. Denote by S' the dual of S, which we call the tempered distributions. We can define the Fourier transform of $l \in S'$ by

$$F(l)(s) = l(F(s)) \ \ \forall s \in S.$$

This in turn allows us to define

$$H^s = \{u \in S' \ \mid \ < \zeta >^s \hat{u}(\zeta) \ \in \ L^2\}$$

An operator

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

is said to be elliptic if when we replace D^α by $\zeta^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and view P as a polynomial in the x_i (i.e. we take the Fourier transform of P), then

$$|L(\zeta)| \geq \lambda |\zeta|^m$$

for $|\zeta| \geq A$, where $|\zeta|$ is the standard norm on n-vectors and A is some constant. $\bar{\partial}$ and Δ are both elliptic: the fact that their kernels consist of smooth functions follows from the theorem we are about to prove, together with the embedding of $H^s \subset C^k$ for $s \gg k$.

We are given an elliptic operator P of order m, which maps H^{s+m} into H^s. We want to prove

$$Lu \in H^s \Rightarrow u \in H^{s+m},$$

i.e. that applying P to a function always kills m Sobolev derivatives.

Better still, we will prove

$$||u||_{s+m} \leq C(||Pu||_s + ||u||_2) \tag{3}$$

for some constant C. This is called elliptic regularity.

$P(D)u \in H^s$ is equivalent to $< \zeta >^s P(\zeta)\hat{u}(\zeta) \in L^2$. Consider

$$\int < \zeta >^{2(s+m)} |\hat{u}(\zeta)|^2 d\zeta = \int_{|\zeta| \leq A} < \zeta >^{2(s+m)} |\hat{u}(\zeta)|^2 d\zeta + \int_{|\zeta| > A} < \zeta >^{2(s+m)} |\hat{u}(\zeta)|^2 d\zeta$$

We have

$$\int_{|\zeta| \leq A} < \zeta >^{2(s+m)} |\hat{u}(\zeta)|^2 d\zeta < C \int |\hat{u}(\zeta)|^2 d\zeta$$

where

$$C = \sup_{|\zeta| \leq A} < \zeta >^{2(s+m)} .$$

We bound the second term by

$$
\begin{aligned}
\int_{|\zeta|>A}\int <\zeta>^{2(s+m)} |\hat{u}(\zeta)|^2 d\zeta &\leq \int_{|\zeta|>A} <\zeta>^{2s}<\zeta>^{2m} |\hat{u}(\zeta)|^2 d\zeta \\
&\leq \frac{1}{\lambda}\int_{|\zeta|>A} <\zeta>^{2s} |P(\zeta)|^2|\hat{u}(\zeta)|^2 d\zeta \\
&= \frac{1}{\lambda}\int_{|\zeta|>A} <\zeta>^{2s} |\widehat{P(D)u}(\zeta)|^2 d\zeta = \frac{1}{\lambda}||Pu||_s.
\end{aligned}
$$

This proves (3). The analogous statement on the level of distributions is as follows: define the singular support of a distribution u as by

$$
\text{singsupp}(u)^c = \{x \in \mathbb{R}^n : \exists \phi \in C^\infty, \phi(x) \neq 0, \phi u \in C^\infty\}.
$$

Then if P is elliptic,

$$
\text{singsupp} u = \text{singsupp} Pu.
$$

These results can be extended to manifolds using a "local" version of Fourier transforms: this is the field of microlocal analysis.

1.4 Summary of a paper of Uhlenbeck

(Michael McBreen)

We summarize part of [19]. The aim is to prove that for certain families of elliptic operators on a compact manifold (possibly with boundary) parametrized by Banach spaces, the following properties hold for residual[3] sets of operators:

1. The spectrum is simple.

2. The eigenfunctions are Morse functions when restricted to the interior of M (i.e. they have a finite number of critical points, and the Hessian is non-degenerate at each critical point).

3. Zero is not a critical value for the eigenfunctions restricted to the interior of M (i.e. the nodal sets are submanifolds).

4. Zero is not a critical value of $\frac{d\phi}{dn}$ on ∂M, where $\frac{d\phi}{dn}$ is the derivative of the eigenfunction ϕ in the normal direction to the boundary.

[3]A set is residual if it is the countable intersection of open dense sets. In complete metric spaces, residual sets are dense by the Baire category theorem.

These results are proved with transversality theory. We'll need the following results:

Sard Smale Theorem: Let $f : M \to N$ be a Fredholm map of class C^l, where $l > max(0, index(f))$ and M, N are separable Banach manifolds. Then the regular (i.e. non-critical) values of f are residual in N.

Transversality Theorem 1: Let $\psi : H \times B \to E$ be a C^k map with $0 \in E$ as a regular value, where H, E, B are Banach manifolds and H, E are separable. If $\psi(-, b)$ is Fredholm of index $< k$, then there is a residual set of $b \in B$ such that $\psi(-, b)$ has 0 as a regular value.

Transversality Theorem 2: Let $f : Q \times X \to Y$ be transverse to $Y' \subset Y$, where X, Y are finite dimensional and Q is separable. Suppose f is C^k with $k > max(1, dimX + dimY' - dimY)$, and suppose that there is a C^k map $\pi : Q \to B$ of Fredholm index zero, where B is a separable Banach manifold. Then the set of $b \in B$ such that $f_b = f|_{\pi^{-1}(b)}$ is transverse to Y' is residual in B.

Sard-Smale tells us that regularity (or transversality) is generic. The other two theorems say that if the evaluation map of a family of operators is transverse to something, then most of the operators are transverse too.

Now, pick a family of degree 2 elliptic operators L_b where $b \in B$, a Banach space. Let $W^{k,p}$ be the space of functions on M whose k first weak derivatives are p-integrable, with the Sobolev norm

$$||f||_{k,p} = \int_M |\nabla^k f|^p + \ldots + |f|^p.$$

Let $W_0^{k,p}$ be the closure of the smooth functions of compact support on the interior of M in $W^{k,p}$. Then we fix some $k \geq 1$ and consider

$$L_b : W^{k,p} \cap W_0^{1,p} \to W^{k-2,p}.$$

Restricting the domain to $W_0^{1,p}$ amounts to imposing Dirichlet conditions (of a sort) on the boundary.

The theory of elliptic operators tells us that if the coefficients of L_b are C^k, then for $\lambda \in \mathbb{R}$,

$$L_b + \lambda : W^{k,p} \cap W_0^{1,p} \to W^{k-2,p}$$

is Fredholm of index 0. [4]

Uhlenbeck proves the following in [19]: Let

[4]Note that since the inclusion $W^{k,p} \subset W^{k-2,p}$ is compact, so is $\lambda : W^{k,p} \to W^{k-2,p}$. Since the Fredholm property and the index are invariant under compact perturbations, it suffices to show the claim for one choice of λ.

$$S^{k,p} = \{u \in W^{k,p} \cap W_0^{1,p}, \int_X |u|^2 dvol = 1\}.$$

Let

$$\phi : S^{k,p} \times \mathbb{R} \times B \to W^{k-2,p}$$

be given by

$$\phi(u, \lambda, b) = (L_b + \lambda)u.$$

Then ϕ is also Fredholm of index zero.

Clearly, $(u, \lambda, b) \in \phi^{-1}(0)$ iff u is an eigenfunction of L_b with eigenvalue λ. We claim that λ is simple[5] iff (u, λ) is a regular point of $\phi_b = \phi(-, -, b)$.

To prove this, consider the linearization of ϕ_b:

$$D\phi_b(\delta u, \delta\lambda) = (L_b + \lambda)\delta u + \delta\lambda u.$$

Here $\delta u \in TS^{k,p}$ iff $\int_M u\delta u dvol = 0$. The image of $(L_b + \lambda)$ is the L_2 orthogonal complement of the eigenspace for λ. Hence $D\phi_b$ is surjective iff u spans the eigenspace.

It follows that L_b has a simple spectrum iff 0 is a regular value for ϕ.

If $\phi : S^{k,p} \times \mathbb{R} \times B \to W^{k-2,p}$ is C^2 and has 0 as a regular value, then it follows from Transversality theorem 1 that for a residual set of b, ϕ_b has zero as a regular value. Hence L_b has simple spectrum for a residual set of b.

So we've reduced the problem to showing that for a given family of operators, ϕ has zero as a regular value. The linearization of ϕ has the form

$$D\phi(\delta u, \delta\lambda, \delta b) = (L_b + \lambda)\delta u + \delta\lambda u + D_b\phi(\delta b)$$

where $D_b\phi$ is the restriction of the linearization of ϕ to the tangent space of B. Showing that zero is a regular value will therefore come down to showing that for all eigenfunctions u and all $\rho \in W^{k-2,p}$, there exists δb, $\delta\lambda$ and δu satisfying

$$\int u\delta u dvol = 0$$

and such that

$$(L_b + \lambda)\delta u + \delta\lambda u + D_b\phi(\delta b) = \rho.$$

[5] i.e. λ corresponds to a one-dimensional eigenspace

Uhlenbeck verifies this for certain common families of operators.

Similarly, one proves that the nodal sets of a residual set of L_b are manifolds by examining the evaluation map

$\Phi : W^{k,p} \times M \to \mathbb{R}, \quad (u, m) \to u(m)$.

Note that this is only defined for $kp > n$, which ensures that $W^{k,p} \subset C^0$. We restrict Φ to the set of eigenfunctions $E = \phi^{-1}(0)$. Assuming that zero is a regular value of ϕ, E is a Banach submanifold of $S^{k,p} \times \mathbb{R} \times B$, and $\Phi(u, \lambda, b, m) = u(m)$. If $\Phi(u, -)$ has zero as a regular value, then the nodal set is a submanifold of M by the implicit function theorem.

There is a projection map $\pi : E \to B$, which can be checked to be Fredholm of index zero. If the spectrum of L_b is simple, then eigenfunctions for that b are isolated from one another and it suffices to check that $\Phi|_{\pi^{-1}(b)} \subset E$ has zero as a regular value. Transversality theorem 2 says that if zero is regular for $\Phi|_E$, it's regular for $\Phi|_{\pi^{-1}(b)}$ for a residual set of b, and the intersection of these with the residual set of 'simple spectrum' b will be residual and hence dense.

Once again, Uhlenbeck checks this for particular families of operators.

1.5 Domain types

We can learn much about the domain Ω of an elliptic operator by studying the billiard flow on its unit sphere bundle $S(T\Omega)$. For simplicity we will take $\dim(\Omega) = 2$. The billiard flow Φ_t takes a unit vector (x, v) to $(\gamma_{x,v}(t), w)$ where $\gamma_{x,v}(t)$ is the geodesic satisfying $\gamma(0) = x$ and $\gamma'(t)|_{t=0} = v$ and w is the parallel transport of v. If the geodesic hits $\partial\Omega$, it is 'reflected' at an angle equal to its angle of incidence.

A domain is said to have ergodic billiards if this flow is ergodic, i.e. if any invariant set in $S(T\Omega)$ has either full measure or measure 0. Examples of ergodic billiards are a square domain with a circle removed (Sinai billiards), or a generic polygon (see Zelditch and Zworski [22]). One can check that geodesic flow on the the flat torus or the torus with metric

$$ds^2 = (f(x) + g(y))(dx^2 + dy^2)$$

is non-ergodic (for the standard torus this is clear, as geodesics are simply projections of straight lines on $\mathbb{R}^2$).

On the other hand, a domain is said to be integrable if it is foliated by invariant subsets of dimension one or zero.

A smooth closed curve in Ω is called a caustic if any billiard trajectory tangent to Ω remains tangent to it after each reflection off the boundary. For instance, an ellipse $\Omega \subset \mathbb{R}^2$ has infinitely many caustics, namely the smaller confocal ellipses and the hyperbolas through Ω. A domain with a continuous family of caustics isn't ergodic, because the set of trajectories tangent to the family of caustics is invariant and will have positive measure. In general, one expects a convex surface with a smooth boundary to have many caustics. In fact, Lazutkin [10] showed that if $\partial\Omega$ is sufficiently differentiable and has curvature bounded away from zero and infinity, then there is a one-parameter family of curves c_t, $t \in [0,1]$ converging to the boundary and a set $A \subset [0,1]$ such that for $t \in A \subset [0,1]$, c_t is a caustic, and $\frac{\mu(A\cap[0,\epsilon])}{\epsilon} \to 1$ as $\epsilon \to 0$.

However, there exist ergodic convex domains, such as the Bunimovich stadium, which looks like a rectangle with rounded corners and two strictly convex sides.

1.6 Random wave conjectures

The length scale for eigenfunction behavior is $\frac{1}{\lambda}$, so for very large λ, the large scale structure of the manifold becomes less important. For instance, the effect of the boundary on the behavior of eigenfunctions on the interior of M diminishes according to the distance from the boundary, measured in wavelengths.

In [4], Michael Berry conjectured that high energy (large λ) eigenfunctions have the same properties as random Fourier series. These conjectures generally fail for "nice" manifolds with many symmetries.

One domain with interesting eigenfunctions to which the conjecture can be applied is $\mathbb{H}$, the upper half plane with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. $SL(2,\mathbb{Z})$ acts on $\mathbb{H}$ by isometries

$$z \to \frac{az+b}{cz+d},$$

which are generated by the operations $T(z) = z+1$ and $S(z) = \frac{-1}{z}$, corresponding to translation and 'inversion' of the half-plane. One can show by elementary arithmetic that the set

$$D = \{x+iy, -\frac{1}{2} < x \leq \frac{1}{2}, x^2+y^2 \geq 1\}$$

is a fundamental domain for this action. The quotient $\mathbb{H}/SL(2,\mathbb{Z})$ can be given a complex structure which makes it isomorphic to $\mathbb{C}$. The eigenfunctions of the Laplacian on the quotient space have deep number-theoretic significance: see Sarnak [14].

Let ϕ be one such eigenfunction with $\Delta\phi = (\frac{1}{4}+t^2)\phi$.

If we expand

$$\phi(x,y) = \sum_{n\neq 0} f_n(y) e^{w\pi i n x}$$

then according to the conjecture $f_n(y) = c_n K_{it}(2y|n|)$ where $K_{it}(2y|n|)$ are Bessel functions and the c_n are distributed normally.

In general, if you have a sequence of functions satisfying some mild assumptions and look at successive random linear combinations of the first n terms, then certain properties of these sums are universal, i.e. don't depend on the particular sequence of functions. For instance, it is known that the maximum of the random linear combinations grows as $\sqrt{log(n)}$. We also know $\forall \epsilon > 0, \quad ||\psi_\lambda||_\infty < \lambda^\epsilon$. Does perhaps $||\psi_\lambda||_\infty \sim (log(\lambda))^k$ for $k = \frac{1}{2}$ or $\frac{1}{4}$? See Kahane [7] for more details.

On a flat torus, the eigenfunctions are actually bounded. Perhaps one can prove that if a completely integrable manifold has bounded eigenfunctions, then that manifold is a torus.

There are known cases when the above conjectures fail. For instance, certain manifolds have "Heegner points" where most of the eigenfunctions of Δ vanish (see [15]). However, by the local version of Weyl's law,

$$\sum_{\sqrt{\lambda_j}\leq\lambda} \phi^2 \sim c\lambda^n + O(\lambda^{n-1}).$$

This implies that some of the eigenfunctions which don't vanish at the Heegner points take large values there, contradicting the bounds on random series. For generic metrics, this (hopefuly) shouldn't happen.

Suppose M is compact or $K \subset \mathbb{R}^n$ is compact. Let

$$f_\lambda(a,b) = \frac{Vol\{x \in M \;\; s.t. \;\; \phi_\lambda(x) \in [a,b]\}}{Vol(M)}, \quad \int_M \phi^2 dvol = 1.$$

It would follow from the conjectures of Berry that after a suitable normalisation,

$$\lim_{\lambda\to\infty} f_\lambda(a,b) = \int_a^b e^{-\frac{x^2}{c}} dx.$$

It turns out that this is equivalent to

$$\int \phi_\lambda^k \to \mu_k$$

where μ_k is the kth moment of the Gaussian distribution (under certain regularity conditions, if you know all the moments of a distribution, you know the distribution itself). In particular, for odd k the integal tends to zero, since the Gaussian is even. This clearly holds for $k = 1$, since

$$\int \phi_\lambda^1 =< \phi_\lambda, 1 >= 0$$

since ϕ_λ and 1 are eigenfunctions of Δ with distinct eigenvalues (at least for Neumann boundary conditions).

The conjecture seems to hold on the modular domain for special classes of eigenfunctions and is known for $k = 3, 4$. It has been tested on supercomputers: see [3].

1.7 The distribution of eigenvalues of $\Delta + tV$

Let M be a compact negatively curved surface, and $V \in C^\infty(M)$, $f \in C_0^\infty([0, \infty])$. Let λ_i be the eigenvalues of Δ and μ_i the eigenvalues of $\Delta + V$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(\lambda_{j+1} - \lambda_j)$$

exists iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(\mu_{j+1} - \mu_j)$$

does, and if so, they are equal. For higher dimensional manifolds, one must take $\lambda_k^{\frac{2}{n}}$ instead, so that Weyl's law gives $\frac{\lambda_k^{\frac{2}{n}}}{k} \to$ constant.

Proof for surfaces: Let $\lambda(t)_j$ be an eigenvalue of $\Delta + tV$, $t \in [0, 1]$. Let $x_j(t) = \lambda_{j+1}(t) - \lambda_j(t)$. It is enough to show

$$\frac{1}{N} \sum_{j=1}^{N} [f(x_j(t)) - f(x_j)] \to 0.$$

We have

$$\left| \frac{d}{dt} \sum_{j=1}^{N} f(x_j(t)) \right| = \left| \sum_{j=1}^{N} f'(x_j(t)) \times [\lambda'_{j+1}(t) - \lambda'_j(t)] \right| \le c \sum_{j=1}^{N} \left| \lambda'_j(t) \right|$$

For each t, there is an orthonormal basis of eigenvectors $\{\psi_j(t)\}$ such that

$$\lambda_j' = \int_M V\psi_j^2(t)dvol.$$

Hence our bound becomes

$$\left|\frac{d}{dt}\sum_{j=1}^{N} f(x_j(t))\right| \leq \frac{c}{N}\sum_{j=1}^{N}\left|\int_M V\psi_j^2(t)dvol\right|.$$

Without loss of generality, we can take $\int_M V dvol = 0$. We now quote the quantum (or semi-classical) ergodicity theorem: if the geodesic flow on M is ergodic, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left|\int_M V\psi_j^2(t)dvol\right| = 0.$$

This concludes the proof.

Along the same lines, we have the following theorem from [21] : Let M be a smooth, compact, negatively curved surface. The the following are equivalent:

1. There exists a limiting distribution of $\lambda_{i+1}(\Delta) - \lambda_i(\Delta)$.[6]
2. There exists a limiting distribution of $\lambda_{j+1}(\Delta + V) - \lambda_j(\Delta + V)$.

For instance, on a sphere the limiting distribution is $\delta(x)$.

We verify quantum unique ergodicity for the circle. $M = S^1$ has ergodic flow if we restrict our attention to right movers. The eigenfunctions of the laplacian are

$$\phi_n(x) = \frac{sin(nx)}{\sqrt(2)}$$

and

$$\psi_n(x) = \frac{cos(nx)}{\sqrt(2)}.$$

We will consider only the $\phi_n(x)$; the other case is identical.

For $V \in C^\infty(S^1)$, $\int V = 0$, we have

$$\int_{S^1} V\psi_n^2(t)dx = \frac{1}{2}\int_{S^1} V sin(nx)^2(t)dx = \int_{S^1} V(1 - cos(2nx))d\theta \xrightarrow{n} 0$$

The last limit is simply the Riemann-Lebesgue lemma. This proves quantum ergodicity for the circle.

[6]In dimension n one should consider $\lambda_{i+1}^{\frac{2}{n}} - \lambda_i^{\frac{2}{n}}$, because $\#\{\lambda_j < \lambda\} \sim c\lambda^{\frac{n}{2}}$.

2 Geodesics on Negatively Curved Manifolds

2.1 Geodesic flow on negatively curved manifolds

The geodesic flow on a Riemannian manifold (M, g) is defined as follows. Define a flow on TM by

$$\psi_t : (x, v) \to (exp_x(tv), dexp_x(tv)(v)),$$

i.e. the map that takes the basepoint to the time t image of the geodesic based at x with initial velocity v. This map is not in general continuous near the zero section (small vectors pointing in opposite directions get sent to very different places) but if we restrict it to the sphere bundle SM associated to TM, i.e. the set of length-one vectors, it becomes a diffeomorphism.

For example, we can look at the geodesic flow on the compact quotient of the hyperbolic plane

$$\mathbb{H}^n = \{x_1, \ldots, x_n | x_n > 0; ds^2 = \frac{\Sigma_i dx_i^2}{x_n^2}\},$$

which has constant curvature $K = -1$, by a subgroup $\Gamma \subset SL(2, \mathbb{R})$:

$$\mathbb{H}^2/\Gamma = [PSL(2, \mathbb{R})/SO(2)]/\Gamma.$$

Since $SL(2, \mathbb{R})$ acts by isometries, we can use the hyperbolic metric on $\mathbb{H}$ to define a metric on the quotient. The unit sphere bundle of $\mathbb{H}^2/\Gamma$ is just $PSL(2, \mathbb{R})/\Gamma$. The geodesic flow on $PSL(2, \mathbb{R})$ is given by left multiplication by

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

We now use the geodesic flow to show why the universal cover of a negatively curved manifold is $\mathbb{R}^n$.

The rate at which geodesics diverge from each other (or converge) is measured by Jacobi vector fields. Let $\gamma_0(t)$ be part of a one-parameter family of geodesics $\gamma_u(t)$, not necessarily with the same endpoints. Then $J(t) = \frac{d\gamma_u(t)}{du}|_{u=0}$ is a vector field along $\gamma(t)$ satisfying

$$\nabla_t^2 J(t) + R\left(\frac{d\gamma(t)}{dt}, J(t)\right) J(t) = 0$$

where $R(-,-)$ is the Riemann curvature tensor and ∇ is the Levi-Civita connection. (see [12], Part III). Conversely, any such vector field corresponds to a family of geodesics. If the norm of J increases along γ, then the geodesics in the corresponding family diverge from γ as t increases. If J vanishes at some point, then the family intersects γ there.

The exponential map $exp_x : T_xM \to M$ takes a tangent vector v at x to the endpoint of the geodesic with initial velocity v. If geodesics with distinct angles starting at x don't intersect, then this map is a covering of M by T_xM. We can sometimes use the Jacobi field associate to the family of geodesics parametrized by the initial angle to determine whether intersections occur. For instance, in two dimensions, the relevant Jacobi fields satisfy an equation roughly of the form

$$J(t)'' + KJ(t) = 0, \quad J(0) = 0, \quad J'(0) = 1$$

where K is the curvature (a scalar). For constant positive curvature, we then get $J(t) \approx \frac{sin(\sqrt{K}t)}{\sqrt{K}}$, which vanishes infinitely often. For negative curvature, $J(t) \approx \frac{sinh(\sqrt{K}t)}{\sqrt{K}}$, which only vanishes at the origin. We expect the exponential map to be injective everwhere except at Jacobi conjugate points[7], i.e. points x, y where $\gamma(0) = x, \gamma(t) = y$ $J(t) = 0$). By the above, there are no conjugate points in negative curvature, hence the exponential map realizes the tangent space at x as the universal cover of M.

One can use the universal cover to find a bijection between conjugacy classes in $\pi_1(M)$ and closed geodesics: see [6].

2.2 Heat and wave traces

Let $\phi_j(x)$ and $\{\lambda_j\}$ be the eigenfunctions and eigenvalues of Δ on a manifold. The fundamental solution of the heat equation $(\frac{\partial}{\partial t} - \Delta)\phi = 0$ on the manifold is

$$\sum_j e^{-t\lambda_j}\phi_j(x)\phi_j(y).$$

We define the heat trace

$$\sum_j e^{-t\lambda_j} = Tr(e^{t\Delta}).$$

We can study this function's behavior as $t \to 0$ to learn about the number of eigenvalues in some range. The coefficients of the asymptotic expansion as

[7]To complete this argument, one must show that geodesics who intersect twice are linked by a continuous family of geodesics intersecting at the same points.

$t \to 0^+$ are called heat invariants. It so happens that the zeroth order term only cares about the volume and dimension of M. Other terms are usually integrals of curvature polynomials. See [5] for more details.

The fundamental solution of the wave equation $(\frac{\partial^2}{\partial t^2} - \Delta)\phi = 0$ is

$$\sum_j cos(t\sqrt{\lambda_j})\phi_j(x)\phi_j(y).$$

Just as with the heat equation, we define the "wave trace" as

$$\sum_j cos(t\sqrt{\lambda_j}) = Even(Tr(e^{it\sqrt{\Delta}})).$$

There is an obvious singularity at $t = 0$, and additional singularities at $t =$ length of a closed geodesic. In a certain sense, geodesic loops at x control the local wave kernel at x. This is due to Helton, Collin de Verdier, Duistermat, Guillemin, and on the physics side, Gutzwiller.

2.3 Separation of geodesics in phase space

We want to understand and prove (a special case of) the statement 'closed trajectories of hyperbolic dynamical systems (eg. geodesic loops on manifolds of negative curvature) are separated in phase space.

Take the two dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ as an example. We consider geodesic flow on the unit sphere bundle of T^2 with the standard flat metric inherited from $\mathbb{R}^2$. The tangent bundle of T^2 is trivial: it inherits a global coordinate system from the tangent bundle of $\mathbb{R}^2$. The unit sphere bundle of T^2 is therefore T^3.

To study geodesics on T^2 it is useful to look at its universal cover $\mathbb{R}^2$, which is tiled by units squares corresponding to different fundamental domains for the quotient map $\mathbb{R}^2 \to T^2$. A geodesic $\gamma(t)$ in T^2 is simply a straight line on the unit square, and it can be lifted to a straight line l in $\mathbb{R}^2$. Suppose $\gamma(0) = (0,0)$ (where we are identifying T^2 with the square $[0,1] \times [0,1]$. Then γ closes on itself iff l intersects the corner of another unit square with integer coordinates. The angle θ of l is therefore $arctan(\frac{m}{n})$, where m, n are relatively prime integers.

We call a closed geodesic primitive if it doesn't loop twice around its image. The number of primitive geodesics of length $l(\gamma) \leq T$ starting at $(0,0)$ is

$$\#\{\gamma : l(\gamma) < T\} \approx \frac{\pi T^2}{\zeta(2)}$$

where πT^2 corresponds to the number of pairs (m,n) such that $m^2+n^2 \leq T^2$, and $\zeta(2)$ compensates for the non-primitive geodesics counted this way (which correspond to points (m,n) where m and n aren't relatively prime).

Clearly, if we have one closed geodesic, then we get a one-dimensional family of them by moving the starting point around on the torus while keeping the angle fixed. We will show that these families are isolated from each other, in the following sense:

There exists $c > 0$ such that the elements of

$$G_T = \{\theta = arctan(\frac{m}{n}) \mid \sqrt{m^2+n^2} \leq T\}$$

are separated by gaps of size $\geq \frac{c}{T^2}$.

Proof: Given two pairs m,n and $m^{'},n^{'}$ satisfying the conditions defining T, we have $|\frac{m}{n} - \frac{m^{'}}{n^{'}}| = |\frac{mn^{'}-m^{'}n}{nn^{'}}| \geq \frac{1}{T^2}$. Now,

$$\frac{d}{dx}arctan(x) = \frac{1}{1+x^2}.$$

Hence for $m/n < 1$, the spacing between consecutive $\theta \in G_T$ is at least half the spacing between consecutive fractions. By symmetry, this holds for $m/n > 1$ too. The claim follows.

The result can be extended to higher dimensions.

2.4 Counting closed geodesics on negatively curved manifolds

One can try to count the number of closed geodesics geodesics of length $\leq T$ on a closed negatively curved manifold M. This problem was solved by Huber for constant negative curvature and by Margulis [11] for variable negative curvature. They found (counting all geodesics, not only primitive ones)

$$\#\{\gamma : L(\gamma) \leq T\} \sim C\frac{e^{hT}}{hT}$$

where h is the *topological entropy* of the geodesic flow on M, and depends on the metric. This is explained as follows: for a point $u \in U$ in the universal cover of M (the hyperbolic half-space), the volume of a ball $B(u,T)$ of radius T centered at u grows as e^{hT}. The number of closed geodesics starting at the projection of u will be roughly proportional to the number of fundamental domains covered by $B(u,T)$, i.e. proportional to its volume.

When $K \equiv -1$, the topological entropy is $n-1$. For variable K we only have $-K_1^2 \leq K \leq -K_2^2 \Rightarrow (n-1)K_2 \leq h \leq (n-1)K_1$.

2.5 Entropy rigitidy

Besson, Courtois and Gallot proved that if M_1 and M_2 are negatively curved, $curv(M_1) \equiv -1$, $vol(M_1) = vol(M_2)$ and entropy(M_1) = entropy(M_2), then $curv(M_2) \equiv -1$. Entropy is therefore quite a rigid property.

Suppose M has a negatively curved metric. How does the entropy vary as the metric varies? Negatively curved manifolds of dimension greater than 2 are rigid: deforming the metric while preserving the curvature yields an isometric manifold. In dimension 2, on the other hand negatively curved metrics on a surface of genus g live in a Teichmuller space T_g of dimension $6g-6$ for $g \geq 2$ and of dimension 2 and 0 for $g = 1$ and $g = 0$ respectively. Katok proved that in dimension 2, the space of metrics of constant curvature is a local extremum for entropy.

2.6 Separating geodesic lengths

The following "spaghetti lemma" is proved in [18]: define a c-spaghetti neighbourhood of a closed geodesic γ of length T as

$$C_\gamma \equiv \{(x, \xi) \in SM, distance_{SM}[(x, \xi), \gamma] \leq e^{-cT}\}$$

Then there is a positive $c > h$ s.t. the C_γ are pairwise disjoint as γ runs over all closed geodesics of length less than T.

One can use the spaghetti neighbourhoods to perturb the metric so that a given pair of geodesics have different lengths (simply use a bump deformation within the disjoint neighbourhoods).

It is known that for generic metrics, all geodesics have different lengths. Contrast this with the constant curvature case in dimensions 2 and 3, where there are always geodesics of the same length.

2.7 Spectral rigidity

Definition: A deformation of the metric g is called trivial if $g_t = \phi_t^* g_0$, where ϕ_t is a smooth diffeomorphism of the manifold.

The following results were published in [8] and [16].

Theorem 1: If M is negatively curved and compact, and g_t preserves the spectrum $\sigma(\Delta)$ of the Laplacian, then g_t is trivial.

Theorem 2: If M is negatively curved and compact, and has simple length spectrum, then $\sigma(\Delta + q_1) = \sigma(\Delta + q_2) \Rightarrow q_1 = q_2$ where q_1, q_2 are smooth potentials.

We will outline the proof of Theorem 2. If you know $\sigma(\Delta + q)$ then you know $\Sigma_{L(\gamma)=T} \int_\gamma q$ for all T, using the trace formula.

Now, in constant negative curvature, if $q \in C^\infty(M)$ and $\int_\gamma q = 0$ for all closed geodesics γ, then $q \equiv 0$. (the abundance of geodesics in the hyperbolic plane is inherited by its quotients).[8]

Hence, if all $L(\gamma)$ are distinct, $\int_\gamma q_1 - q_2 = 0$ for all γ implies $q_2 = q_2$. This concludes the proof.

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[8] We can weaken the assumption to have $\int_\gamma q = 0$ for only certain γ, for instance $L(\gamma) \in [T_i, 1 + T_i], T_i \to \infty$. Note that the lemma is false on the sphere: odd functions integrate to zero on all geodesics.

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