Gauss curvature of random metrics

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- (M, g) is a compact surface with a Riemannian metric g.
- Goal: study Gauss curvature K of *random* Riemannian metrics on M.
- *Gauss curvature:* Geometric meaning: as $r \to 0$,

$$\mathrm{vol}(B_M(x_0, r)) = \pi r^2 \left[1 - \frac{K(x_0) r^2}{12} + O(r^4)\right].$$

$K > 0 \Rightarrow$ surface in $\mathbf{R}^3$ is *convex*; volume grows *slower* than in $\mathbf{R}^2$.
$K < 0 \Rightarrow$ surface in $\mathbf{R}^3$ is *concave*; volume grows *faster* than in $\mathbf{R}^2$.

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- **Conformal class:** Metric g_1 is *conformally equivalent* to g_0 if for all $x \in M$ and $U, V \in T_x M$,

$$g_1(x)(U,V) = F(x) \cdot g_0(x)(U,V), \qquad F(x) > 0.$$

  The set of all such metrics is called a *conformal class* $[g_0]$ of g_0.
- **Uniformization theorem:** in every conformal class, there exists a unique metric of constant Gauss curvature K_0. $K_0 > 0$ for $M = S^2$, $K_0 = 0$ for $M = \mathbf{T}^2$, and $K_0 < 0$ for surfaces of genus $\gamma \geq 2$.
- **Gauss-Bonnet theorem:** $\int_M K dA = 2\pi\chi(M)$, where χ is the *Euler characteristic*, $\chi(\textit{sphere with } \gamma \textit{ handles}) = 2 - 2\gamma$, e.g. $\chi(S^2) = 2, \chi(\mathbf{T}^2) = 0$ etc.

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- **Questions:** Assume $M \neq \mathbf{T}^2$, and g_0 has *non-vanishing* curvature K_0. What is the *probability* that a random metric g_1 in the conformal class $[g_0]$ also has non-vanishing curvature K_1?
- Use *Laplacian* to define random metrics in a *conformal class* and to estimate that probability.
- Techniques: differential geometry; spectral theory of Laplacian; Gaussian random fields on manifolds (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).

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- g_0 - reference metric on M. Conformal class of g_0: $\{g_1 = e^f \cdot g_0\}$; f is a random (suitably regular) function on M.
- Δ_0 - Laplacian of g_0. Spectrum: $\Delta_0\phi_j + \lambda_j\phi_j = 0, \;\; 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$. Define f by

$$f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x), \tag{1}$$

where $a_j \sim \mathcal{N}(0,1)$ are i.i.d standard Gaussians, $c_j = F(\lambda_j) \to 0$ are *decreasing*.

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- Functions on $\mathbf{T}^2$: $h(x) = \sum_m c_m e^{i(x,m)}, m \in \mathbf{Z}^2$. Sobolev norm: $(||f||_{H^s})^2 = \sum_m |c_m|^2(1+||m||^2)^s$.
  General surface: $f(x) = \sum_j c_j \phi_j(x)$.
  $||f||^2_{H^s} = \sum_j c_j^2 (1+\lambda_j)^s$.
  Sobolev embedding theorem: If $s > k+1$, and $||f||_{H^s} < \infty$, then $f \in C^k(M)$.
  Weyl's law: $\lambda_j \asymp const \cdot j$.
- *Random functions: f* as in (1), then

$$\mathbb{E}(||f||^2_{H^s}) = \sum_j c_j^2(1+\lambda_j)^s.$$

- **Proposition 1:** If $c_j < C/\lambda_j^s, s > 1$, then $f \in C^0(M)$ a.s; if $c_j < C/\lambda_j^s, s > 2$, then $f \in C^2(M)$ a.s.

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- The *covariance function* $r_f(x,y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y)$, for $x, y \in M$.
- For $x \in M$, $f(x)$ is mean zero Gaussian of variance $r_f(x,x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2$.
- **Area change:** Let $A_0 = \mathrm{area}(M, g_0)$. If $g_1 := g_1(a) = e^{2af} g_0$, then $dA_1 = e^{2af} dA_0$. One can show that $\lim_{a\to 0} \mathbb{E}[\mathrm{area}(M, g_1(a))] = A_0$.

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- Let $g_1 = e^{2af} g_0$. Then

$$K_1 = e^{-2af}[K_0 - a\Delta_0 f] \tag{2}$$

$M \neq \mathbf{T}^2$. Estimate the probability of

$$\{\mathrm{Sgn}(K_1) = \mathrm{Sgn}(K_0)\}$$

- **Observation:** If $K_0 \neq 0$, then $\mathrm{Sgn}(K_1) = \mathrm{Sgn}(K_0)\mathrm{Sgn}(1 - a\Delta_0 f/(K_0))$.
- Let $P(a) := \mathrm{Prob}\{\exists x : \mathrm{Sgn}K_1(x) \neq \mathrm{Sgn}K_0\}$, or $P(a) = \mathrm{Prob}\{\exists x \in M : 1 - a(\Delta_0 f)(x)/K_0(x) < 0\}$. Then

$$P(a) = \mathrm{Prob}\{\sup_{x \in M}(\Delta_0 f)(x)/K_0(x) > 1/a\},$$

Consider the random field $v = (\Delta_0 f)/K_0$. Then

$$r_v(x,y) = \frac{\sum_j (c_j\lambda_j)^2 \phi_j(x)\phi_j(y)}{K_0(x)K_0(y)}.$$

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- We shall estimate $P(a)$ in the limit $a \to 0$. Geometrically, this implies that a.s. $g_1(a) \to g_0$, so $P(a) \to 0$. We want to estimate the *rate*.
- First use **Proposition 2** (Borell, TIS, 1975-76): Let v be a centered Gaussian process, a.s. bounded on M, and $\sigma_v^2 := \sup_{x \in M} \mathbb{E}[v(x)^2]$. Let $||v|| := \sup_{x \in M} v(x)$; then $E\{||v||\} < \infty$, and $\exists \alpha$ so that for $\tau > E\{||v||\}$ we have

$$\mathrm{Prob}\{||v|| > \tau\} \leq e^{\alpha\tau - \tau^2/(2\sigma_v^2)}.$$

- Assume that $K_0 \in C^0, s > 2$, then $v \in C^0(M)$ a.s. and Proposition 2 applies. In our situation, $\tau = (1/a) \to \infty$ as $a \to 0$, so $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma_v^2)]$.

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- To estimate $P(a)$ *from below* choose $x_0 \in M$ where the variance $r_v(x,x)$ attains its supremum σ_v^2. Clearly, $\mathrm{Prob}(||v|| > 1/a) \geq \mathrm{Prob}(v(x_0) > 1/a) = \frac{1}{\sqrt{2\pi}} \int_{1/(a\sigma_v)}^{\infty} e^{-t^2/2} dt$. Combine the estimates:
- **Theorem 3:** Assume that $R_0 \in C^0, c_j = O(\lambda_j^{-s}), s > 2$. Then $\exists C_1 > 0, C_2 > 0$ such that

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- **Random real-analytic metrics.** Choose the coefficients $c_j = e^{-\lambda_j T/2}/\lambda_j$. Then

$$r_v(x,x,T) = e^*(x,x,T)/(K_0(x))^2.$$

where $e^*(x,x,T)$ is the heat kernel, *without the constant term.*

- **Small T asymptotics** of $e^*(x,x,T)$ imply that as $T \to 0^+$,

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- **Theorem 4.** $M \neq \mathbf{T}^2$. Let g_0 and g_1 have equal areas, R_0 and R_1 have constant sign, $K_0 \equiv const$ and $K_1 \not\equiv const$. Then $\exists a_0 > 0$, $T_0 > 0$ (that depend on g_0, g_1) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $P(a, T, g_1) > P(a, T, g_0)$.
- **Proof:** By Gauss-Bonnet, $\int_M K_0 dA_0 = \int_M K_1 dA_1$. Since $\mathrm{A}(M, g_0) = \mathrm{A}(M, g_1)$; and since $K_0 \equiv const$ and $K_1 \not\equiv const$, it follows that $b_0 := \min_{x \in M}(K_0(x))^2 > \min_{x \in M}(K_1(x))^2 := b_1$. Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma_v^2(g_1, T)}{\sigma_v^2(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$

The result follows easily from Theorem 3.

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- **Large T asymptotics:**
  λ_1 - the smallest nonzero eigenvalue of $-\Delta_0$. Let $m = m(\lambda_1)$ be the multiplicity of λ_1, and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^{m} \phi_j(x)^2}{K_0(x)^2}. \tag{3}$$

- One can show that

$$\lim_{T \to \infty} \frac{\sigma_v^2(T)}{F e^{-\lambda_1 T}} = 1.$$

- **Theorem 5.** Let g_0 and g_1 be two metrics (of equal area) on a compact surface M, such that K_0 and K_1 have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on g_0, g_1), such that for all $a < a_0$ and $T > T_0$ we have $P(a, T; g_0) < P(a, T; g_1)$.

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- To summarize: Small $T \Rightarrow$ metrics with $K_0 \equiv const$ extremal.
- Large $T \Rightarrow$ metrics with the largest λ_1 extremal.
- Genus 0: (S^2, *round*) extremal for *both* small T and large T (Hersch). **Conjecture:** extremal for *all* T.
- Genus $\gamma \geq 2$: Small $T \Rightarrow$ hyperbolic metrics extremal.
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- Since Δ_0 on (S^2, g_0) is highly degenerate, we normalize our random Fourier series differently.
- $\mathcal{E}_m$ - space of spherical harmonics of degree m, dimension $N_m = 2m+1$; the corresponding eigenvalue is $E_m = m(m+1)$. Let $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ be an orthonormal basis of $\mathcal{E}_m$.
- Let $f(x) = -\sqrt{|\mathcal{S}^2|} \sum\limits_{m\geq 1,\, k} \frac{\sqrt{c_m}}{E_m\sqrt{N_m}} a_{m,k}\eta_{m,k}(x)$, where $a_{m,k}$ are standard Gaussian i.i.d. and $c_m > 0$ are (suitably decaying) constants satisfying $\sum\limits_{m=1}^{\infty} c_m = 1$.
- It follows that $v = \sqrt{|\mathcal{S}^2|} \sum\limits_{m\geq 1,\, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k}\eta_{m,k}(x)$ has unit variance, and covariance $r_v(x,y) = \sum\limits_{m=1}^{\infty} c_m P_m(\cos(d(x,y)))$, where P_m is the Legendre polynomial.

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- We next estimate the probability of the event $\{||K_1 - K_0||_\infty < u\}, u > 0$; we shall do that for $g_1 = e^{af} g_0$, in the limit $a \to 0$. The result below hold for any compact orientable surface, including $\mathbf{T}^2$.
- To state the result, we define a new random field w on M:

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- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when $a \nrightarrow 0$.
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension $n \geq 3$).
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