Measures on spaces of Riemannian metrics
CMS meeting
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D. Jakobson (McGill), jakobson@math.mcgill.ca

- [CJW]: Y. Canzani, I. Wigman, DJ: arXiv:1002.0030, Jour. of Geometric Analysis, 2013
- [CCKJST]: B. Clarke, N. Kamran, L. Silberman, J. Taylor, Y. Canzani, DJ: arXiv:1309.1348, Ann. Math. du Québec, 2015

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Conjectures in Quantum Chaos: Random Wave conjecture, GOE level spacing conjecture, conjectures about nodal sets etc.

Integration on manifolds of metrics:

D. Ebin defined L^2 metric on manifolds of metrics. Restricted to the space of hyperbolic metrics on a compact surface, we get Weil-Petersson metric on moduli spaces of hyperbolic metrics (singular manifolds!). Many people studied *differentiation* on manifolds of metrics; such notions as Levi-Civita connection, geodesics, parallel transport etc. have been defined.

Want to define integration. Our spaces of metrics are infinite-dimensional, so it is convenient to define Gaussian measures on those spaces.

Fix a compact smooth Riemannian manifold M^n. We shall discuss several measures on manifolds of metrics on M.

• Measures on conformal classes of metrics: concentrated near a reference metric g_0, supported on regular (e.g. Sobolev, real-analytic) metrics a.s. Applications to the study of Gauss curvature.

• Measures on manifolds of metrics with the fixed *volume form*, applications to the study of L^2 (Ebin) distance function, and to integrability of the diameter, eigenvalue and volume entropy functionals.

• **Remark:** All measures are invariant by the action of diffeomorphisms.

Conformal class: g_0 - reference metric on M. Conformal class of g_0: $\{g_1 = e^f \cdot g_0\}$; f is a random (suitably regular) function on M.
Δ_0 - Laplacian of g_0. Spectrum:
$\Delta_0 \phi_j + \lambda_j \phi_j = 0, \;\; 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Define f by

$$f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x),$$

$a_j \sim \mathcal{N}(0,1)$ are i.i.d standard Gaussians, $c_j = F(\lambda_j) \to 0$ (*damping*):

The *covariance function*
$r_f(x,y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y)$, for $x, y \in M$.
For $x \in M$, $f(x)$ is mean zero Gaussian of variance
$r_f(x,x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2$.

Examples:

• Random *Sobolev* metric: $c_j = \lambda_j^{-s}$, $\implies$
$r_f(x,y) = \sum_j \frac{\phi_j(x)\phi_j(y)}{\lambda_j^{2s}}$, spectral zeta function.

• Random *real-analytic* metric $c_j = e^{-\lambda_j t}$, $\implies$
$r_f(x,y) = \sum_j \phi_j(x)\phi_j(y) e^{-2\lambda_j t}$, heat kernel.

Sobolev regularity: If

$$\mathbb{E}||f||^2_{H^s} = \sum_j c_j^2 (1+\lambda_j)^s < \infty$$

then $f \in H^s(M)$ a.s. Weyl's law + Sobolev embedding imply
Proposition: If $c_j = O(\lambda_j^{-s}), s > n/2$, then $f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, then $f \in C^2$ a.s.

• [CJW]: Let $n = 2$, and let g_0 have non-vanishing Gauss curvature ($M \neq \mathbf{T}^2$). Can estimate the probability that after a random conformal perturbation, the Gauss curvature will change sign somewhere on M.
• Techniques: curvature transformation in $2d$ under conformal changes of metrics, large deviation estimates (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).
• $n \geq 3$: related results for scalar curvature and Q-curvature.

- Random (Sobolev) embeddings into $\mathbf{R}^k$: 1-dimensional i.i.d. Gaussians $\to$ k-dimensional i.i.d. Gaussians.
- F. Morgan (1979): $M = S^1, k = 3$: a.s. results about minimal surfaces spanned by random “knots.”
- F. Morgan (1982): general compact M, a.s. Whitney embedding theorems + applications.
- We shall use similar ideas to define Gaussian measures on manifolds of metrics with *fixed volume form*; transverse to conformal classes.

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- Metrics = sections of $\mathrm{Pos}(M) \subset \mathrm{Sym}(M) \subset Hom(TM, T^*M)$ (positive-definite, symmetric maps); symmetric matrices in local coordinates. $\mathrm{GL}(T_xM)$ acts on $\mathrm{Pos}_x(M)$ with stabilizer isomorphic to $\mathrm{O}(n)$.
  Fix a volume form v, consider $\mathrm{Met}_v(M)$. $SL(T_xM)$ acts on the fibre $\mathrm{Pos}_x^v(M)$ by
  $$h.g_x = h^T \circ g_x \circ h;$$
  stabilizer isomorphic to $\mathrm{SO}(n)$. We have
  $$\mathrm{Pos}_x^v(M) \cong \mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n$$
- Fix $g^0 \in \mathrm{Met}_v$; $dv(x) = \sqrt{|\det g^0(x)|}dx_1 \wedge \ldots \wedge dx_n$. Let $G_x = \mathrm{SL}(T_xM), K_x = \mathrm{SO}(g_x^0)$.

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- f_x frame in T_xM orthonormal w.r.to g^0_x, $A_x \subset G_x$ positive diagonal matrices of determinant 1 (w.r. to f_x).
  Every $g^1_x \in \mathrm{Pos}^v_x(M)$ can be represented as

  $$g^1_x = (k_x a_x) g^0_x, \qquad k_x \in K_x, a_x \in A_x;$$

  unique up to S_n acting on f_x.
- Assumption: M is *parallelizable* ($\exists$ global section of the frame bundle). Examples:
  - All 3-manifolds;
  - All Lie groups;
  - The frame bundle of any manifold;
  - The sphere S^n iff $n \in \{1, 3, 7\}$.

  Necessary condition: vanishing of the 2nd Stiefel-Whitney class. For orientable $\Longleftrightarrow$ spin.

- f_x frame in $T_x M$ orthonormal w.r.to g_x^0, $A_x \subset G_x$ positive diagonal matrices of determinant 1 (w.r. to f_x).
  Every $g_x^1 \in \mathrm{Pos}_x^v(M)$ can be represented as

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M parallelizable. Choose a global section f^0 of the frame bundle orthonormal w.r. to g^0.

To define g^1_x, we apply to f^0 a composition of a rotation $k_x \in \mathrm{SO}(T_x M)$ and a diagonal unimodular transformation $a_x \in \mathrm{SL}(T_x M)$ which will define an orthonormal basis f^1_x for g^1_x.

By construction, g^0 and g^1 will have the same volume form.

We let $a_x = \exp(H(x))$, where $H : M \to \mathfrak{a} \cong \mathbf{R}^{n-1}$ is the Lie algebra of $\mathrm{Diag}_0(n) \subset \mathrm{SL}_n$. Similarly, let $k_x = \exp h(x)$, where $h : M \to \mathfrak{so}_n$, the Lie algebra of SO_n. Choice of g^0 + parallelizability makes the above construction well-defined.

We define Gaussian measures on $\{H : M \to \mathfrak{a}\}$ and $\{h : M \to \mathfrak{so}_n\}$ as in Morgan. In the sequel, only need H; constructions are analogous.

Let

$$H(x) = \sum_{j=1}^{\infty} \pi_n(A_j)\beta_j\psi_j(x), \tag{1}$$

where

• $\Delta\psi_j + \lambda_j\psi_j = 0$;
• A_j are i.i.d standard Gaussians in $\mathbf{R}^n$;
• $\pi_n : \mathbf{R}^n \to \{x \in \mathbf{R}^n : x \cdot (1, \dots, 1) = 0\} \simeq \mathbf{R}^{n-1}$ - projection into the hyperplane $\sum_{j=1}^{n} x_j = 0$;
• $\beta_j = F_2(\lambda_j) > 0$, where $F_2(t)$ is (eventually) monotone decreasing function of t, $F(t) \to 0$ as $t \to \infty$.

- **Smoothness:** Morgan showed
  Proposition 1: If $\beta_j = O(j^{-r})$ where $r > (q+\alpha)/n + 1/2$, then H converges a.s. in $C^{q,\alpha}(M, \mathbf{R}^{n-1})$.
- Proposition 1 + Weyl's law $\Longrightarrow$
  Proposition 2: If $\beta_j = O(\lambda_j^{-s})$ where $s > q/2 + n/4$, then H converges a.s. in $C^q(M, \mathbf{R}^{n-1})$.

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Lipschitz distance ρ:

$$\rho(g_0, g_1) = \sup_{x \in M} \sup_{0 \neq \xi \in T_x M} \left| \ln \frac{g_1(\xi, \xi)}{g_0(\xi, \xi)} \right| \tag{2}$$

A related expression appeared in the paper by Bando-Urakawa. If $g_1(x) = (k(x)d(x))g_0(x)$ then ρ only depends on the diagonal part $d(x)$.

Tail estimate for ρ:

One can show that for large R,

$$\mathrm{Prob}\{\rho(g_0, g_1) > R\} \leq 2^n(n + \epsilon) \cdot \mathrm{Prob}\{\sup_{x \in M} d_1(x) > R/2\} \tag{3}$$

Definition of d_1:

Recall from (1): $H(x) = \sum_{j=1}^{\infty} \pi_n(A_j)\beta_j\psi_j(x)$.

Define $D(x) = (d_1(x), \ldots, d_n(x))$ by

$$D(x) = \sum_{j=1}^{\infty} A_j\beta_j\psi_j(x)$$

("don't project A_j").

The covariance function for $d_1(x)$:

$$r_{d_1}(x,y) = \sum_{k=1}^{\infty} \beta_k^2\psi_k(x)\psi_k(y),$$

Define σ^2 by

$$\sigma^2 := \sigma(d_1)^2 := \sup_{x\in M} r_{d_1}(x,x). \tag{4}$$

Borell-TIS theorem applied to the random field d_1 implies

Proposition 3. Let σ be as in (4). Then

$$\lim_{R\to\infty} \frac{\ln \mathrm{Prob}\{\rho(g_0, g_1) > R\}}{R^2} \leq \frac{-1}{8\sigma^2}. \tag{5}$$

More precise result:

Proposition 4. There exists $\alpha > 0$ such that for a fixed $\epsilon > 0$ and for large enough R, we have

$$\mathrm{Prob}\{\rho(g_1, g_0) > R\} \leq 2^n(n+\epsilon)\exp\left(\frac{\alpha R}{2} - \frac{R^2}{8\sigma^2}\right).$$

ρ controls diameter and eigenvalues:
Proposition 5.
Assume that $d\mathrm{vol}(g_0) = d\mathrm{vol}(g_1)$ and $\rho(g_0, g_1) < R$. Then

$$e^{-R} \leq \frac{\mathrm{diam}(M, g_1)}{\mathrm{diam}(M, g_0)} \leq e^{R} \tag{6}$$

and

$$e^{-2R} \leq \frac{\lambda_k(\Delta(g_1))}{\lambda_k(\Delta(g_0))} \leq e^{2R}. \tag{7}$$

Propositions 4 and 5 imply

Theorem 6.

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function such that for some $\delta > 0$

$$h(e^y) = O\left(\exp\left[y^2(1/(8\sigma^2) - \delta)\right]\right).$$

Then $h(\mathrm{diam}(g_1))$ is integrable with respect to the probability measure $d\omega(g_1)$ constructed earlier.

and

Theorem 7. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function such that for some $\delta > 0$

$$h(e^{2y}) = O\left(\exp\left[y^2(1/(8\sigma^2) - \delta)\right]\right).$$

Then $h(\lambda_k(\Delta(g_1)))$ is integrable with respect to the probability measure $d\omega(g_1)$ constructed earlier.

Similar results can be established for *volume entropy*,

$$h_{vol} = \lim_{s \to \infty} \frac{\ln \mathrm{vol} B(x, s)}{s}$$

- L^2 or *Ebin* distance between can be computed as follows [Ebin, Clarke]:

$$\Omega_2^2(g^0, g^1) := \int_M d_{2,x}(g^0(x), g^1(x))^2 dv(x)$$

where $d_{2,x}(g^0(x), g^1(x))$ is the distance in $\mathrm{SL}_n/\mathrm{SO}_n$;

$$= \int_M \langle H(x), H(x) \rangle_{g^0(x)} dv(x).$$

- For us Ω_2^2 is a *random variable* whose distribution we shall compute. Note: only depends on H, hence it suffices to consider the Gaussian measure on $\{H : M \to \mathfrak{a}\}$.

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In local coordinates, let

$$a_x = \mathrm{diag}(\exp(b_1(x)), \exp(b_2(x)), \ldots, \exp(b_n(x))),$$

where $\sum_{j=1}^{n} b_j(x) = 0, \forall x \in M$. Then

$$d_x(g_x^0, g_x^1)^2 = \sum_{j=1}^{n} b_j(x)^2.$$

Accordingly,

$$\Omega_2(g^0, g^1)^2 = \int_M \left(\sum_{j=1}^{n} b_j(x)^2 \right) dv(x).$$

$\pi_n : \mathbf{R}^n \to \{x : \sum x_j = 0\}$ standard projection. P_n - matrix of π_n (in the usual basis of $\mathbf{R}^n$) with singular values

$$(1, \ldots, 1, 0) := \mu_{i,n}, 1 \leq i \leq n.$$

Then in distribution

$$\Omega_2^2 \stackrel{D}{=} \sum_j \beta_j^2 \sum_{i=1}^{n} \mu_{i,n}^2 W_{i,j}$$

where $W_{i,j} \sim \chi_1^2$ are i.i.d. We get $\Omega_2^2 \stackrel{D}{=} \sum_j \beta_j^2 V_j$ where $V_j \sim \chi_{n-1}^2$ are i.i.d.

Theorem 8.
Moment generating function of Ω_2^2:

$$M_{\Omega_2^2}(t) = E(\exp(t\Omega_2^2)) = \prod_j \prod_{i=1}^{n} M_{\chi_1^2}(t\mu_{i,n}^2\beta_j^2)$$

$$= \prod_j \prod_{i=1}^{n} (1 - 2t\mu_{i,n}^2\beta_j^2)^{-1/2} = \prod_j (1 - 2t\beta_j^2)^{-(n-1)/2}$$

Characteristic function of Ω_2^2:

$$\prod_j \prod_{i=1}^{n} (1 - 2it\mu_{i,n}^2\beta_j^2)^{-1/2} = \prod_j (1 - 2it\beta_j^2)^{-(n-1)/2}$$

Corollary 9.
Tail estimates for Ω_2^2: applying results of Laurent-Massart, one can show that

$$\mathrm{Prob}\{\Omega_2^2 \geq s^2\} \leq \exp(-s^2/(2\beta_1^2)).$$