Math 581, Winter 2018 D. Jakobson

PROBLEM SET 1, PART 2 Due date to be announced

Do all the problems. Every problem is worth 5 points. Some problems will not be graded because of time constraints.

GA 3. Let $u(x)$ be a homogeneous polynomial of degree $k \geq 0$ in $\mathbb{R}^n$, that is

$$u(x) = \sum_{i_1+\ldots+i_n=k} A_{i_1\ldots i_n} x_1^{i_1} \ldots x_n^{i_n}, \quad and \quad \Delta u = 0.$$

where $i_j \in \mathbb{Z}, i_j \geq 0$. Let v be the restriction of $u(x)$ so the surface of the unit sphere $S^{n-1} \in \mathbb{R}^n$. Show that v is an eigenfunction of the Laplacian for the round metric on S^{n-1} with the eigenvalue $\lambda = k(k+n-2)$.

Hint: Using Problem **GA 2**, write $\Delta_{\mathbb{R}^n}$ in spherical coordinates.

SP 1. Spectral radius. Let E be a Banach space, and let T be a bounded linear operator on E. Let $a_n = \log ||T^n||, n \geq 1$.

(a) Check that $a_{i+j} \leq a_i + a_j$.

(b) Deduce that $\lim_{n\to\infty}(a_n/n)$ exists and coincides with $\inf_{m\geq 1}(a_m/m)$. Hint: Fix $m \geq 1$. Given $n \geq 1$, write $n = mq + r$ (divide n by m with remainder r and $q = \lfloor n/m \rfloor$). Note that $a_n \leq (n/m)a_m + a_r$.

(c) Conclude that $r(T) = \lim_{n\to\infty} ||T^n||^{1/n}$ exists and that $r(T) \leq ||T||$. $r(T)$ is called the *spectral radius* of T. Construct an example with $E = \mathbb{R}^2$ such that $r(T) = 0$ and $||T|| = 1$.

SP 2.

Let $E = L^p(0,1), 1 \leq p \leq \infty$. Define the operator T by the formula

$$Tu(t) = \int_0^t u(s)ds.$$

(a) Prove by induction that for $n \geq 2$,

$$(T^n u)(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} u(\tau) d\tau$$

(b) Deduce that $||T^n|| < 1/n!$ Hint: use the inequality for the convolution product

(c) Prove that $r(T) = 0$. Hint: use Stirling's formula.