1. Measurability and Fubini’s Theorem

Recall that for measurable \( f : \mathbb{R}^2 \to \mathbb{R} \), its sections \( f_x \) and \( f_y \) are defined by
\[
f_x(y) = f_y(x) := f(x, y).
\]

Let \((X, S)\) and \((Y, T)\) be measurable spaces with their corresponding \(\sigma\)-algebras; as usual, we denote by \(S \times T\) the smallest \(\sigma\)-algebra on \(X \times Y\) containing all the rectangles.

The following proposition follows easily from the measurability property of sections of measurable sets in \(S \times T\):

**Proposition 1.1.** Let \( f : X \times Y \to \mathbb{R} \) be \(S \times T\)-measurable. Then \( f_x \) is \(T\)-measurable and \( f_y \) is \(S\)-measurable.

The analogues of Theorems 1.10 and 1.12 in Lieb/Loss are proved next, using the definition if the integral through simple functions.

2. Completion of Product Measures

Let \(L_k\) denote the Lebesgue measure on \(\mathbb{R}^k\). Let \(L_m \times L_k\) denote the smallest \(\sigma\)-algebra on \(\mathbb{R}^m \times \mathbb{R}^k\) containing all rectangles. That measure is not complete.

**Proposition 2.1.** The completion of \(L_m \times L_k\) is \(L_{m+k}\).

Fubini’s theorem for completions of measures:

**Theorem 2.2.** Let \((X, S, \mu)\) and \((Y, T, \lambda)\), and let \((S \times T)^*\) be the completion of \(S \times T\), relative to the measure \(\mu \times \lambda\). Let \( f \) be \((S \times T)^*\)-measurable on \(X \times Y\). Then

i) If \( 0 \leq f \leq \infty \), then \( \phi(x) := \int_Y f_x \ d\lambda \) is defined for \(\mu\)-a.e. \( x \); \( \psi(y) := \int_X f_y \ d\mu \) is defined for \(\lambda\)-a.e. \( y \); \( \phi \) is \(S\)-measurable, \( \psi \) is \(T\)-measurable, and
\[
\int_X \phi \ d\mu = \int_Y \psi \ d\lambda = \int_{X \times Y} f \ d(\mu \times \lambda).
\]

ii) If \( f : X \times Y \to \mathbb{C} \), let \( \phi^*(x) = \int_Y |f_x|^\lambda \ d\lambda \). Assume \( \int_X \phi^* \ d\mu < \infty \). Then \( f \in L^1(\mu \times \lambda) \), i.e. \( \int_{X \times Y} |f|^\lambda \ d(\mu \times \lambda) < \infty \).

iii) If \( f \in L^1(\mu \times \lambda) \), then \( f_x \in L^1(\lambda) \) for \( \mu\)-a.e. \( x \in X \); \( f_y \in L^1(\mu) \) for \(\lambda\)-a.e. \( y \in Y \). The function \( \phi \) defined in (i) is in \(L^1(\mu)\), and the function \( \psi \) defined in (i) is in \(L^1(\lambda)\), and (1) holds.