MATH 564, FALL 2009. LEBESGUE INTEGRATION: SUMMARY OF THE MATERIAL IN RUDIN

1. Measurability and Fubini's Theorem

Recall that for measurable $f : \mathbf{R}^2 \to \mathbf{R}$, its *sections* f_x and f_y are defined by

$$f_x(y) = f_y(x) := f(x, y).$$

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces with their corresponding σ-algebras; as usually, we denote by $\mathcal{S} \times \mathcal{T}$ the smallest σ-algebra on $X \times Y$ containing all the rectangles.

The following proposition follows easily from the measurability property of sections of measurable sets in $\mathcal{S} \times \mathcal{T}$:

Proposition 1.1. *Let $f : X \times Y \to \mathbf{R}$ be $\mathcal{S} \times \mathcal{T}$-measurable. Then f_x is $\mathcal{T}$-measurable and f_y is $\mathcal{S}$-measurable.*

The analogues of Theorems 1.10 and 1.12 in Lieb/Loss are proved next, using the definition if the integral through simple functions.

2. Completion of Product Measures

Let $\mathcal{L}_k$ denote the Lebesgue measure on $\mathbf{R}^k$. Let $\mathcal{L}_m \times \mathcal{L}_k$ denote the smallest σ-algebra on $\mathbf{R}^m \times \mathbf{R}^k$ containing all rectangles. That measure is *not* complete.

Proposition 2.1. *The completion of $\mathcal{L}_m \times \mathcal{L}_k$ is $\mathcal{L}_{m+k}$.*

Fubini's theorem for completions of measures:

Theorem 2.2. *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$, and let $(\mathcal{S} \times \mathcal{T})^*$ be the completion of $\mathcal{S} \times \mathcal{T}$, relative to the measure $\mu \times \lambda$. Let f be $(\mathcal{S} \times \mathcal{T})^*$-measurable on $X \times Y$. Then*

i) *If $0 \leq f \leq \infty$, then $\phi(x) := \int_Y f_x \, d\lambda$ is defined for μ-a.e. x; $\psi(y) := \int_X f_y \, d\mu$ is defined for λ-a.e. y; ϕ is $\mathcal{S}$-measurable, ψ is $\mathcal{T}$-measurable, and*

$$\int_X \phi d\mu = \int_Y \psi d\lambda = \int_{X \times Y} f d(\mu \times \lambda). \tag{1}$$

ii) *If $f : X \times Y \to \mathbf{C}$, let $\phi^*(x) = \int_Y |f|_x d\lambda$. Assume $\int_X \phi^* d\mu < \infty$. Then $f \in L^1(\mu \times \lambda)$, i.e. $\int_{X \times Y} |f| d(\mu \times \lambda) < \infty$.*

iii) *If $f \in L^1(\mu \times \lambda)$, then $f_x \in L^1(\lambda)$ for μ-a.e. $x \in X$; $f_y \in L^1(\mu)$ for λ-a.e. $y \in y$. The function ϕ defined in (i) is in $L^1(\mu)$, and the function ψ defined in (i) is in $L^1(\lambda)$, and (1) holds.*