Math 564, Fall 2011 D. Jakobson

SKETCHES OF SOLUTIONS, HOME MIDTERM

Problem 1. Define the measure on $[0,1]$ by

$$\mu([a,b)) = \log_2 \frac{1+b}{1+a}.$$

a) Prove that μ is preserved by the map $f(x) = \{1/x\}$, where $\{y\}$ denotes the fractional part of y.

Every real number in $x \in [0,1]$ can be expanded into a (finite or infinite) *continued fraction*

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \ldots}}}, \tag{1}$$

sometimes denoted by $x = [n_1, n_2, n_3, \ldots]$.

b) Prove that finite continued fractions correspond to rational numbers, while infinite fractions correspond to irrational numbers.

c) Prove that the function f in part a) can be written as a *shift map*,

$$f([n_1, n_2, n_3, \ldots]) = [n_2, n_3, \ldots].$$

Solution: Note that $\mu([0,1]) = 1$. We have for $0 < a < b < 1$,

$$f^{-1}([a,b]) = \cup_{n=1}^{\infty} \left(\frac{1}{n+b}, \frac{1}{n+a} \right)$$

The measure μ of that set is equal to

$$\sum_{n=1}^{\infty} \log_2 \left(\frac{(b+n)(a+n+1)}{(a+n)(b+n+1)} \right)$$
$$= \sum_{n=1}^{\infty} [\log_2(b+n) + \log_2(a+n+1) - \log_2(a+n) - \log_2(b+n+1)]$$
$$= \log_2(1+b) - \log_2(1+a) = \mu([a,b]).$$

Since the interval was arbitrary, we are done.

Part (c) is obvious from the definition of f, since for x as in (1),

$$1/x = n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \ldots}}.$$

Clearly, finite continued fractions give rise to rational numbers (clear the denominators). Conversely, if we apply the map f to a rational number $p/q, 0 < p < q$, the result $\{q/p\}$ will have a smaller denominator p, so after $\leq q$ applications of f we shall get 1 and the continued fraction will terminate.

Problem 2. We keep the notation from Problem 1.

a) Describe the measure μ in problem 1 in the space of sequences $[n_1, n_2, n_3, \ldots]$.

b) Describe all the *periodic* continued fractions, $x = [n_1, \ldots, n_k, n_1, \ldots, n_k, \ldots]$.

Solution: Part b): the real number x satisfies the equation

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \ldots + \frac{1}{n_k + x}}}.$$

Clearing the denominators, it is easy to see by induction that x satisfies

$$x = \frac{Ax + B}{Cx + D}$$

where A, B, C, D are integers that depend on $n_1, \dots, n_k$; it follows that x satisfies quadratic equation with integer coefficients, so it is a *quadratic irrational*. In fact, every quadratic irrational gives rise to *eventually periodic* continued fraction, but we won't prove it.

Part a). We want to compute the measure of the cylinder consisting of all continued fractions with $n_1 = a_1, \dots, n_k = a_k$, where $(a_1, \dots, a_k)$ is a given k-tuple of natural numbers. This set corresponds to the interval in $(0, 1)$ with rational endpoints at $\alpha_k = \frac{1}{a_1 + \frac{1}{a_2 + \dots \frac{1}{a_k}}}$ and $\beta_k = \frac{1}{a_1 + \frac{1}{a_2 + \dots \frac{1}{a_k + 1}}}$; the interval will be $[\alpha_k, \beta_k]$ for even k, and $[\beta_k, \alpha_k]$ for odd k. The measure is then computed by the usual formula.

Problem 3. Let $x, y \in [0, 1]$, $x = [0.n_1 n_2 \dots]$ and $y = [0.m_1 m_2 \dots]$. Define $f(x, y) = k$, if $n_k = m_k$, but $n_j \neq m_j$ for $1 \leq j \leq k - 1$; and $f(x, y) = \infty$ if no such k exists. Prove that f is Lebesgue measurable on $[0, 1] \times [0, 1]$, and that it is finite almost everywhere.

Solution: The measure of the set where f is finite is equal to

$$\frac{10}{100} + 90 \left(\frac{10}{10^4} + 90 \left(\frac{10}{10^6} + \dots \right) \right) = 1.$$

Equivalently, the measure of all $(x, y) \in [0, 1]^2$ s.t. the k-th digits in the decimal expansions of x and y never coincide for $1 \leq k \leq n$ is equal to $(9/10)^n \to 0$ as $n \to \infty$.

Now, $f(x, y)$ is measurable since it a sequence of step functions $f_k(x, y) = \min\{k, f(x, y)\}$ converge to f a.e.

Problem 4. Let $f \in L^1(X, \mu)$ and $\mu(X) = 1$. Prove that there exists a monotone function $g \in L^1([0, 1])$, such that for all $t \in [0, 1]$,

$$\inf_{\mu(A)=t} \int_A f(x) d\mu(x) = \int_0^t g(\tau) d\tau$$

$$\sup_{\mu(A)=t} \int_A f(x) d\mu(x) = \int_{1-t}^1 g(\tau) d\tau$$

Hint: consider the function

$$h(a) = \mu(\{x \in X : f(x) \leq a\}).$$

Solution: It is clear that $h(a)$ is nondecreasing, $h(a) \to 0$ as $a \to -\infty$, and $h(a) \to 1$ as $a \to +\infty$. We shall solve the problem in the case when $h(a)$ is continuous. We shall also assume that μ takes all values between 0 and 1 (e.g. that the space X is not discrete); and that the function f is never equal to constant on a set of positive measure. Under those assumptions, the function $h(a)$ is *strictly* monotone, and so there exists a function $g(t)$ defined on $[0, 1]$ s.t. $h(g(t)) = t$. Let us show that $\inf_{\mu(A)=t} \int_A f(x) d\mu(x)$ is attained on $X(t) := \{x : f(x) \leq g(t)\}$. Indeed, let A be any subset of X of measure t; then $\mu(Y) := \mu(X(t) \setminus (X(t) \cap A))$ is equal to $\mu(Z) := \mu(A \setminus (X(t) \cap A))$. But by definition of $X(t)$, the function function f is $\leq g(t)$ on Y, and $\geq g(t)$ on Z. It follows that the set attaining the infimum is uniquely defined (up to a set of measure 0).

We next show that $\mu\{x : f(x) \leq a\} = \mu_0\{t \in (0,1) : g(t) \leq a\}$. The right hand side is equal to $h(a)$, since $g(t) \leq a$ is equivalent to $t \leq h(a)$; the left hand side is equal to $h(a)$ by definition; such functions are called *equimeasurable*. It follows that for any Borel function ξ, we have $\int_0^1 \xi(g(t))dt = \int_X \xi(f(x))d\mu(x)$. In particular, putting ξ to be the characteristic function of $[0,t]$ gives the first identity.

To prove the second identity, we remark that $\sup_{\mu(A)=t} \int_A f(x)dx$ is attained on the set $Y(t) := \{x : f(x) \geq g(1-t)\}$. By definition, $\mu(Y(t)) = 1 - \mu\{x : f(x) \leq g(1-t)\} = 1 - h(g(1-t)) = 1-(1-t) = t$; the proof that the supremum is attained on $Y(t)$ is similar to the proof of the corresponding statement for the infimum. To complete the proof, we choose ξ to be the characteristic function of $[1-t,1]$.

Problem 7. Let $x = a_0 + a_1 p + a_2 p^2 + \ldots \in \mathbf{Z}_p$. We define $||x|| = p^{-k}$, where k is the smallest integer s.t. $a_k \neq 0 \pmod p$. Recall that we have defined a measure $\mu = \mu_p$ on $\mathbf{Z}_p$ (Assignment 2, Part 2, Problem 2), by requiring that $\mu(p^n\mathbf{Z}_p) = p^{-n}$. Let $s > 0$. Compute $\int_{\mathbf{Z}_p} ||x||^{-s}\mu(x)$.

Hint: Let $W_k = \{x \in \mathbf{Z}_p : ||x|| = p^{-k}\}$; those are measurable disjoint sets. Then the integral I satisfies $I = \sum_{k=0}^{\infty} p^{-ks}\mu(W_k)$.

Solution: We need to compute $\mu(W_k)$. We have $W_k = p^k\mathbf{Z}_p \setminus p^{k+1}\mathbf{Z}_p$. By the definition of μ, we find that $\mu(W_k) = p^{-k} - p^{-k-1}$. Accordingly,

$$I = \sum_{k=0}^{\infty} p^{-ks} p^{-k-1}(p-1) = \frac{p-1}{p(1-p^{s-1})} = \frac{p-1}{p-p^{-s}}.$$

Problem 8. Let $f \in L^1(\mu)$. Prove that for each $\epsilon > 0$, there exists $\delta > 0$, such that $\int_E |f|d\mu < \epsilon$ for any measurable E with $\mu(E) < \delta$.

Solution: Find a simple function $g : X \to \mathbf{R}^+, g \leq |f|$, such that $\int |f| - \int g \leq \epsilon/2$. Let $g = \sum_{j=1}^n a_j\chi(E_j), a_j \geq 0$. Suppose $\mu(E) < \delta$.

Now,

$$\int_E |f|d\mu = \int_E (|f|-g)d\mu + \int_E g d\mu \leq \int_X (|f|-g)d\mu + \delta\cdot(\max_j a_j) < \epsilon/2 + \delta\cdot(\max_j a_j).$$

So, it suffices to take $\delta < \epsilon/(2\max_j a_j)$.