Math 480/695, D. Jakobson

Problem Set 2

Due: Thursday, January 25 in class

Problem 1 (circulant matrices). An $n \times n$ matrix A is called a *circulant* matrix if its i-th row is equal to a cyclic shift of its first row by $(i-1)$ steps, i.e. if $a_{i,j} = a_{1,j-i+1}$ (where the subscripts are considered modulo n). It has the form

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_n & a_1 & \dots & a_{n-2} & a_{n-1} \\ & & \dots & & \\ a_3 & a_4 & \dots & a_1 & a_2 \\ a_2 & a_3 & \dots & a_n & a_1 \end{bmatrix}.$$

(i) Let W be the circulant $n \times n$ matrix whose first row is equal to $(0,1,0,\dots,0)$. Prove that the characteristic polynomial of W is equal to $x^n - 1$. Its roots are the numbers
$$1 = \omega^0, \omega, \omega^2, \dots, \omega^{n-1}$$
where ω denotes the (complex) root of unity $\exp(2\pi i/n)$.

(ii) Prove that
$$A = a_1 \cdot \mathrm{Id} + a_2 \cdot W + a_3 \cdot W^2 + \dots + a_n \cdot W^{n-1}.$$
Coclude that the the eigenvalues of A are given by
$$\lambda_k = \sum_{j=1}^{n} a_j \omega^{(j-1)k}, \qquad k = 0, 1, \dots, n-1.$$

(iii) Using (ii), find the eigenvalues of the adjacency matrix and the number of spanning trees for the following graphs: the complete graph K_n; the cycle C_n; the *hyperoctahedral* graph H_s obtained by removing s disjoint edges from K_{2s} (also called the *cocktail-party* graph); the *Möbius ladder* M_s obtained from the cycle C_{2s} by connecting s pairs of "opposite" vertices (it is a cubic graph).

Problem 2. Biggs, Chapter 8, p. 178, # 17.

Problem 3. The vertices of the *line graph* $L(G)$ of the graph G are the edges of G. Two vertices of $L(G)$ are connected iff the corresponding edges have a common vertex. Let the vertices of G be $v_1, \dots, v_n$ and let its edges be $e_1, \dots, e_m$. Form an $n \times m$ matrix $\mathbf{X} = (\mathbf{X})_{ij}$ where $\mathbf{X}_{ij} = 1$ if v_i and e_j are incident and $\mathbf{X}_{ij} = 0$ otherwise.

(i) Prove that $(\mathbf{X}^t)\mathbf{X} = A(L(G)) + 2 \cdot \mathrm{Id}_m$.

(ii) Prove that if G is k-regular, then $\mathbf{X}\mathbf{X}^t = A(G) + k \cdot \mathrm{Id}_n$.

(iii) Prove that any eigenvalue λ of $A(L(G))$ satisfies $\lambda \geq -2$.

Problem 4.

(i) Prove that dual graphs have the same number of spanning trees.

(ii) Using Euler's formula $V - E + F = 2$ (where V is the number of vertices, E is the number of edges and F is the number of faces of a convex polyhedron), prove that there exist only five regular polyhedra.

Problem 5. A *strongly regular* graph with parameters (k, a, c) is a k-regular graph where every pair of *adjacent* vertices has a common neighbours and every pair of *non-adjacent* vertices has c common neighbours. Prove that the adjacency matrix A of a such a graph satisfies

$$A^2 + (c - a)A + (c - k)\mathrm{Id} = cJ,$$

where J is the matrix of 1-s.

Problem 6. Biggs, Chapter 8, p. 178, # 21.