Math 454, Fall 2018 D. Jakobson

PROBLEM SET 1, corrected **Due date to be announced**

Do any 8 of the following problems. Every problem is worth 10 points.

Problem 1. Royden/Fitzpatrick, Chapter 2, Problem 7. A set of real numbers is called a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there exists a G_δ set G for which $E \subseteq G$ and $m^*(G) = m^*(E)$.

Problem 2. Royden/Fitzpatrick, Chapter 2, Problems 17 and 18.

(a) Show that E is measurable iff for each $\epsilon > 0$, there exists a closed set F and an open set O s.t. $F \subseteq E \subseteq O$, and $m^*(O \setminus F) < \epsilon$.

(b) Let E be a measurable set that has a finite outer measure. Show that there exists an F_σ set F and a G_δ set G s.t. $F \subseteq E \subseteq G$, and $m^*(F) = m^*(E) = m^*(G)$.

Problem 3. Royden/Fitzpatrick, Chapter 2, Problem 22. For any set A, define $m^{**}(A) \in [0, \infty]$ by

$$m^{**}(A) = \inf\{m^*(O) : A \subseteq O, O - \mathit{open}\}.$$

How is the set function m^{**} related to the outer measure m^*?

Problem 4. Royden/Fitzpatrick, Chapter 2, Problem 26. Let $\{E_k\}$ be a countable disjoint collection of measurable sets. Prove that for any set A,

$$m^*\left(A \cap (\cup_{k=1}^{\infty} E_k)\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

Problem 5. Royden/Fitzpatrick, Chapter 2, Problem 28. Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

Problem 6. Royden/Fitzpatrick, Chapter 2, Problem 32. Does Lemma 16 remain true if Λ is allowed to be finite or to be uncountably infinite? Does it remain true if Λ is allowed to be unbounded?

Problem 7. Royden/Fitzpatrick, Chapter 2, Problem 34. Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure 0.

Problem 8. Royden/Fitzpatrick, Chapter 2, Problem 37. Let F be a continuous function defined on E. Is it true that $f^{-1}(A)$ is always measurable if A is measurable?

Problem 9. Royden/Fitzpatrick, Chapter 2, Problem 44. A subset $A \subseteq \mathbb{R}$ is called *nowhere dense* in $\mathbb{R}$ provided that every open set O, there exists an open $U \subseteq O$ s.t. $U \cap A = \emptyset$. Show that the (middle thirds) Cantor set is nowhere dense in $\mathbb{R}$.

Problem 10 (extra credit). Royden/Fitzpatrick, Chapter 2, Problem 41. A nonempty subset X of $\mathbb{R}$ is called *perfect* provided it is closed, and each neighbourhood of any point on X contains infinitely many points in X. Show that the (middle-thirds) Cantor set $\mathbf{C}$ is perfect. Hint: consider the endpoints of all subintervals defined in the construction of the Cantor set.

Problem 11 (extra credit). Royden/Fitzpatrick, Chapter 2, Problem 42. Prove that every perfect set is uncountable.

Problem 12 (extra credit). Prove that $\mathbf{C} + \mathbf{C} := \{x + y : x, y \in \mathbf{C}\} = [0, 2]$.