McGill University

Math 355: Honors Analysis 4 — Winter 2011

Practice Problems

1. Prove that for the double integral

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dxdy$$

both repeated integrals exist, but that they are not equal. Why is there no contradiction with Fubini's theorem?

2. Verify Lusin's theorem for the function $f(x) = \arcsin(1/x^2), x \in (0,1]$.

3. Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt \qquad (x > 0)$$

to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} dx \;=\; \frac{\pi}{2}.$$

4. For which values of α and β, the function $f(x) = x^\alpha (\sin x)^\beta$ is Lebesgue integrable on $(0,1]$?

5. Let $f \in C^1([a,b])$, and let $f(a) = f(b) = 0$. Show that

$$\int_a^b |f(x)|^2 dx \le \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 dx.$$

Hint: Without loss of generality (by rescaling), we may assume that $f \in C^1([0,1/2])$; extend f to $[-1/2, 0]$ by the formula $f(-x) = -f(x)$, and extend to $\mathbf{R}$ by periodicity. Show that the resulting function is in $C^1(\mathbf{T}^1)$, and apply the Parseval identity.

6. For $n \ge 0$, let

$$f(x,y) = \begin{cases} 2^{2n}, & 2^{-n} \le x \le 2^{-n+1}, 2^{-n} \le y < 2^{-n+1}; \\ -2^{2n+1}, & 2^{-n-1} \le x \le 2^{-n}, 2^{-n} \le y < 2^{-n+1}; \\ 0, & \text{otherwise.} \end{cases}$$

Show that iterated integrals exist but are not equal to each other.

7. Give an example of a sequence of functions $f_n : [0,1] \to \mathbf{R}$ such that

a) $f_n(x) \to 0$ as $n \to \infty$, for all $x \in [0,1]$.

b) f_n converges strongly to 0 in $L^1([0,1])$;

c) f_n does not converge strongly to 0 in $L^2([0,1])$.

8. Consider the triangle Δ in $L^2([0,1])$ with vertices at the functions $1, x, 6x^2$. Find the angle at the vertex 1, and determine the center and the radius of the circumscribed circle of Δ.

9. Let $f \in L^2([-\pi,\pi])$, and $f(x+\pi) = f(x)$. What can you say about Fourier coefficients of f?

10. Prove that the set of points at which a sequence of measurable real-valued functions converges to a finite limit is measurable.

11. Let $0 < \alpha \le \beta < \infty$. For which values of p the function $f(x) = 1/(x^\alpha + x^\beta)$ belongs to $L^p((0,\infty), dx)$?

12. Let $1/p+1/q+1/r=1$, where $p,q,r>0$. Let $f \in L^p(X,\mu), g \in L^q(X,\mu), h \in L^p(X,\mu)$. Prove that $fgh \in L^1(X,\mu)$ and that

$$||fgh||_1 \le ||f||_p \cdot ||g||_q \cdot ||h||_r.$$

13. Let $f \in L^\infty([0,1]), ||f||_\infty > 0$. Let $\alpha_n = \int_0^1 |f(x)|^n dx$. Show that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = ||f||_\infty.$$

14.

a) Let $1<p<\infty$, and let $f,g \in L^p(X,\mu), ||f||_p = ||g||_p = 1, f \ne g$. Prove that $||(f+g)/2||_p < 1$; in other words, the unit ball in $L^p(X,\mu)$ is *strictly convex.*

b) Prove that a) fails in $L^1(X,\mu)$.

15. Prove that $\lim_{a\to\infty} \int_0^a (\sin(x)/x)dx$ exists, but that $\sin(x)/x$ is not Lebesgue integrable on $[0,\infty)$.

16. Prove that a system of *Haar functions*

$$\phi_{m,n}(x) = \begin{cases} 2^{m/2}, & x \in [(n-1)/2^m, (n-1/2)/2^m], \\ -2^{m/2}, & x \in [(n-1/2)/2^m, n/2^m], \\ 0, & x \notin [(n-1)/2^m, n/2^m]. \end{cases}$$

forms an orthonormal basis of $L_2([0,1],dx)$.

17. Let D_N denote the N-th Dirichlet kernel, $D_N(t) = \sum_{n=-N}^{N} e^{int}$. Prove that its L^1 norm satisfies

$$||D_N||_1 \ge c \log N.$$

18. Compute the Fourier series of the function

$$f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & 1 < |x| \le \pi. \end{cases}$$

19. Let $f \in L^1(\mathbf{T})$, and $m>0$. Let $g(t) := f(mt)$. Show that

$$\hat{g}(m) = \begin{cases} \hat{f}(n/m), & if\ m|n; \\ 0, & if\ m \not| n. \end{cases}$$

20 (optional). Let $f \in L^1(\mathbf{T}), g \in L^\infty(\mathbf{T})$. Show that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} f(x)g(nx)dx = \hat{f}(0)\hat{g}(0).$$

21 (optional). Let $f \in L^1(\mathbf{T})$. Define the operator $T: L^1(\mathbf{T}) \to L^1(\mathbf{T})$ by $T(g) = f * g$. Show that $||T||_op = ||f||_{L^1}$. Hint: take $g_n = K_n$ to be the n-th Fejer kernel.

22. Let $f \in \mathrm{Lip}_\alpha(\mathbf{T}), 0<\alpha<1$. Show that

$$|\sigma_n(f,x) - f(x)| \le C/n^\alpha,$$

where $\sigma_n(f,x) = (S_0(f,x) + \ldots + S_n(f,x))/(n+1) = (f * K_n)(x)$ denotes the convolution of f with the n-th Fejer kernel. Recall that $f \in \mathrm{Lip}_\alpha(\mathbf{T})$ if $|f(x+y) - f(x)| < K|y|^\alpha$, for some fixed K and all $|y| \le \pi$.

23 (optional). f is *analytic* on $\mathbf{T}$ if $f \in C^\infty(\mathbf{T})$ and there exists $0 < R < \infty$ such that $\sup_x |f^{(n)}(x)| \le n! \cdot R^n$. You can assume without proof that such f can be represented by convergent Taylor series. Show that there exists $a > 0, K > 0$ such that

$$|\hat{f}(n)| \le K e^{-a|n|}.$$