Analysis III
Theorems, Propositions & Lemmas... Oh My!

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Proposition 1. *If* $\underline{x} = (x_1, x_2, \ldots), \underline{y} = (y_1, y_2, \ldots)$*, then*

$$d(\underline{x}, \underline{y}) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p \right)^{1/p}$$

is a distance.

Proposition 2. *In the space of continuous functions on* $[a, b]$*,*

$$d(f, g) = \left[\int_a^b |f(x) - g(x)|^p dx \right]^{1/p}$$

defines a distance. We recall the L_p *norm to be*

$$||f||_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

Proposition 3. $||x - y||_p$ *defines a* p*-adic distance on* $\mathbb{Q}$*.*

Proposition 4.

$$||x|| := \sqrt{(}x, x)$$

always defines a norm.

Lemma 5. *The following identity holds.*

$$(x, y) \leq ||x|| \cdot ||y||$$

Proposition 6. *Let* $(a_1, \ldots, a_n, \ldots)$ *be such that*

$$\sum a_i^2 < \infty$$

and let $(b_1, \ldots, b_n, \ldots)$ *be such that*

$$\sum b_j^2 < \infty$$

and let

$$||\underline{a}|| = \left(\sum_{i=1}^{\infty} a_i^2 \right)^{1/2}$$

Then

$$\sum_{i=1}^{\infty} a_i b_i \leq ||a|| \cdot ||b||$$

and so

$$(\underline{a}, \underline{b}) = a_1 b_1 + \cdots + a_n b_n \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{j=1}^{n} b_j^2} \to ||a|| \cdot ||b||$$

as $n \to \infty$.

Lemma 7. *Let* $a > 0, b > 0$, *and* $p, q \geq 1$ *and such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

NB: We say that p, q *are* ***Conjugate Exponents***. *Then*

$$(*) \qquad a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Theorem 8. *Holder's Inequality*
Let $p < 1$

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1$$

Theorem 9. *Minkowski Inequality (For Sequences With* $n < \infty$*)*
Recall that for $\underline{x} = (x_1, \ldots, x_n)$, *we have*

$$||\underline{x}||_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}$$

Proposition 10. *Let $x \in X$ be a contact point. Then, x must be one of the following.*

- *x is an isolated point $x \in A$*
- *x is a limit point of A, $x \in A$*
- *x is a limit point of A, $x \notin A$*

Also, if x is not an isolated point if and only if $\exists (x_n)_{n=1}^{\infty}$ where the x_n are distinct such that $x_n \to x$ as $n \to \infty$ and thus

$$\lim_{n\to\infty} d(x, x_n) = 0$$

Proposition 11.

$$\overline{\overline{A}} = \overline{A}$$

Proposition 12. *(i) $A_1 \subset A_2 \Longrightarrow \overline{A_1} \subset \overline{A_2}$*

(ii) $A = A_1 \cup A_2 \Longrightarrow \overline{A} = \overline{A_1} \cup \overline{A_2}$

Proposition 13. *X_∞ with l_∞ distance is not separable.*
Look at $A \subset X$, $A = \{$ all infinite sequences of 0's and 1's$\}$. A is not countable by the Cantor Diagonalization Argument.

Lemma 14. *If $x_\alpha \neq x_\beta$, then $B(x_\alpha, 1/3) \bigcap B(x_\beta, 1/3) = \emptyset$.*

Proposition 15. *Let $(A_\alpha)_{\alpha\in I}$ be a collection of closed sets. Then*

$$B = \bigcap_{\alpha\in I} A_\alpha$$

is also closed.

Proposition 16. *Let $A_1, \ldots, A_n$ be closed. Then*

$$B = \bigcup_{i=1}^{n} A_i$$

is closed.

Proposition 17. *A is open if and only if $X - A$ is closed.*

Proposition 18. *If A_α is open for any $\alpha \in I$, then*

$$B = \bigcup_{\alpha\in I} A_\alpha$$

is also open.

Proposition 19. *A finite intersection of open sets is also open. That is, if A_i is open for $i = 0, \ldots, n$, then*

$$\bigcap_{i=1}^{n} A_i$$

is also open. Beware, however, that this is not necessarily true for the infinite case. An example of this would be that if

$$A_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) \subset \mathbb{R}$$

which is clearly open for any n, then

$$B = \bigcap_{n=1}^{\infty} A_n = [0, 1]$$

which is closed.

Lemma 20. *$\{G_\alpha\}$ forms a basis if and only if for any open set A, and for any $x \in A$, there exists α such that*

$$x \in G_\alpha \subset A$$

Proposition 21. *Let X be a metric space. We claim that X is second countable if and only if X is separable.*

Corollary 22. *Open sets with respect to d_1, d_2 are the same.*

Corollary 23. *The topologies (space X together with a collection of open sets) defined by d_1, d_2 are the same.*

Proposition 24. *Distances in $\mathbb{R}^n$ defined by*

$$d_p(\underline{x}, \underline{y}) = \left(\sum_{j=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

for $p \geq 1$

$$d_\infty(\underline{x}, \underline{y}) = \max_{1 \leq j \leq n} |x_i - y_i|$$

are equivalent.

Corollary 25. *If a topological space is not Hausdorff, then it is not metrizable.*

Proposition 26. *$f : X \to Y$ is continuous at every point $x \in X$ if and only if for every open set U in Y,*

$$f^{-1}(U) \subset X$$

is also open. Moreover, this is also equivalent to saying that for every closed set B in Y,

$$f^{-1}(B) \subset X$$

is also closed. Another way to say this is

$$X - f^{-1}(B) = f^{-1}(Y - B)$$

Proposition 27. *Points in K are in one to one correspondence with*

$$\sum_{j=1}^{\infty} \frac{a_j}{3^j}$$

such that $a_j \in \{0, 2\}$.

Theorem 28. *If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then*

$$g \circ f : X \to Z$$

is also continuous.

Theorem 29. *The space $(C([a, b]), d_\infty)$ is complete.*

Theorem 30. *Let $1 \leq p < \infty$. Then $(C([a, b]), d_p)$ is not complete.*

Theorem 31. *The space X is complete if and only if for every sequence of nested closed balls*

$$\cdots \subset B(x_n, r_n) \subset \cdots \subset B(x_1, r_1)$$

such that $r_n \to 0$ as $n \to \infty$. We should have a nonempty intersection.

Proposition 32. *Equivalence is an equivalence relation.*

Proposition 33.

$$d = 0 \Longleftrightarrow (x_n) \sim (y_n)$$

Theorem 34. *(Contraction Mapping Principle)*
If X is complete, and A is a contraction mapping, then there exists a unique point $x_0 \in X$ such that $A(x_0) = x_0$ is a fixed point of A. Moreover, for any $x \in X$, $A^n(x) \to x_0$ as $n \to \infty$.

Theorem 35. *Let X be complete, and suppose that $A \subset X$. A is sequentially compact if and only if A is totally bounded.*

Proposition 36. *X is complete, and $A \subseteq X$ is compact if and only if for all $\epsilon > 0$, there exists in X a sequentially compact ϵ-net that covers A.*

Theorem 37. *(Arzela)*
Let $\mathcal{F}$ be a family of continuous functions on $[a, b]$. Then $\mathcal{F}$ is sequentially compact in $(C([a, b]), d_\infty)$ if and only if $\mathcal{F}$ is uniformly bounded and equicontinuous.

Proposition 38. *Let $A \subseteq X$. If A is sequentially compact, then A is sequentially compact in itself if and only if A is as a closed subset of X.*

Corollary 39. *Any closed bounded subset of $\mathbb{R}^n$ is sequentially compact in itself.*

Proposition 40. *Let X be a metric space. X is a compactum if and only if X is complete and totally bounded.*

Proposition 41. *Every compactum has a countable dense subset.*

Theorem 42. *The following are equivalent:*

(i) *X is a compactum.*

(ii) *An arbitrary open cover $\{U_\alpha\}_{\alpha \in I}$ of X has a finite subcover. That is there exists $\alpha_1, \ldots, \alpha_n \in I$ such that*

$$X \subseteq \bigcap_{i=1}^{n} U_{\alpha_i}$$

(iii) *(Finite Intersection Property)*
A family $\{F\}_{\alpha \in I}$ of closed subsets of X such that every finite collection of F_n's has a nonempty intersection has

$$\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$$

Theorem 43. *If X is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact.*

Theorem 44. *If X is compact and $f : X \to Y$ is continuous, one-to-one and onto, then f^{-1} is also continuous and hence f is also a homeomorphism.*

Theorem 45. *Let $D \subseteq C(X,Y)$ where X, Y are compact. Then D is compact in $C(X,Y)$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$,*

$$d_X(x_1, x_2) < \delta \Longrightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

for every $f \in D$.

Theorem 46. *A closed subset Y of a compact set X is compact.*

Proposition 47. *Let X be a metric space. Then Y is sequentially compact in X if and only if $\overline{Y}$ is compact in itself.*

Proposition 48. *Any function that is continuous on a compact metric space X is uniformly continuous.*

Proposition 49. *If X is compact, then $f : X \to \mathbb{R}$ attains a least upper bound U and greatest lower bound L.*

Q: How do you catch a tiger in the desert?
A: First, check if you're in Nevada. If so, then you can just google Mike Tyson's home address and catch it there. If not, then divide the desert in half, each time checking which half the tiger isn't in, then divide the complement of that half in half and so on. Then, once you're left with a segment of desert just big enough for you and the tiger... nab him with an appropriately sized ϵ-net!