Math 354, Honors Analysis, D. Jakobson

Short summary: things to review and remember

A *distance* (p, q) between points p, q in a metric space satisfies

- $\text{dist}(p, q) > 0$ if $p \neq q$; $\text{dist}(p, p) = 0$.
- $\text{dist}(p, q) = \text{dist}(q, p)$.
- $\text{dist}(p, q) + \text{dist}(q, r) \geq \text{dist}(p, r)$.

Main examples of metric spaces (in this course):

- $\mathbf{R}^n$, intervals.
- Spaces of sequences: l_p, l_∞.
- Spaces of functions: $C([0,1])$ with distances $d_p, p < \infty$ (incomplete), d_∞ (complete).
- Space of bounded functions $B(X, Y)$, with the uniform (supremum) distance. It is complete if Y is complete.
- (optional). The space $\mathcal{CL}(\mathcal{X}, \mathcal{Y})$ of continuous linear functionals from a Banach space X to a Banach space Y, with the operator norm. It is complete.

A sequence p_k converges to p in a metric space X iff for any $\varepsilon > 0$ there exists a natural number N such that for every $k > N$, $\text{dist}(p_k, p) < \varepsilon$. A sequence p_k is *Cauchy* iff for any $\varepsilon > 0$ there exists a natural number N such that for every $k, l > N$, $\text{dist}(p_k, p_l) < \varepsilon$. A sequence in $\mathbf{R}^n$ converges iff all the coordinate sequences converge.
Let X, Y be metric spaces, and let f be a function from X to Y. f is *continuous at* $x \in X$ iff one of the following two equivalent conditions holds:

- for any sequence $x_k \to x$ in X, $f(x_k) \to f(x)$ in Y;
- for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\text{dist}_X(x', x) < \delta$ then $\text{dist}_Y(f(x'), f(x)) < \varepsilon$.

The function f is *continuous* iff one of the following three equivalent conditions holds:

- f is continuous at every point of X;
- for any open set $U \subset Y$, $f^{-1}(U)$ is open in X.
- for any closed set $V \subset Y$, $f^{-1}(V)$ is closed in X.

A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is continuous iff all its coordinate functions are continuous.
Things to review about $\mathbf{R}^n$:

- inner products and norm in $\mathbf{R}^n$; Cauchy-Schwarz inequality; Minkowski and Hölder inequalities.
- different definitions of distance give rise to the same open sets in $\mathbf{R}^n$.

Some useful facts about $\mathbf{R}^n$:

- a subset of $\mathbf{R}^n$ is *compact* iff it is *closed* and *bounded*.
- a ball in $\mathbf{R}^n$ is *convex*.
- an *open* subset of $\mathbf{R}^n$ is connected iff it's path connected (not true for arbitrary subsets of $\mathbf{R}^n$).

Things to remember about open and closed sets:

- a set A is open iff every point x in A is an *interior point* of A, i.e. if x has a neighborhood (an open ball centered a t x) which is contained in A.

- a set A is closed iff every *limit point* x of A (a limit of a sequence of points in A) is contained in A;
- A is open iff its complement is closed;
- arbitrary union of open sets is open, arbitary intersection of closed sets is closed;
- *finite* intersection of open sets is open, *finite* union of closed sets is closed;
- the empty set and the whole space are both open and closed.

Things to remember about interior, exterior, and boundary:

- a point x is in the *interior* of A (denoted $\text{Int}A$) iff it's an interior point of A;
- a point x is in the *exterior* of A (denoted $\text{Ext}A$) iff it's an interior point of the complement A;
- a point x is on the *boundary* of A (denoted $\text{Bd}A$) iff it's neither an interior nor an exterior point of A, i.e. if in every neighborhood of x there are points from A and the complement of A;
- $\text{Int}A, \text{Ext}A$ and $\text{Bd}A$ are disjoint; their union is the whole space;
- A is open iff $A = \text{Int}A$, B is closed iff $\text{Bd}B \subseteq B$.

Let f, g be continuous functions into $\mathbf{R}$, and let a, b be real numbers. Then $af + bg, fg$ are continuous, and f/g is continuous provided $g \neq 0$. Also, if $f : X \to Y$ is continuous, $g : Y \to Z$ is continuous on $f(X)$, then their composition $g(f(x))$ is a continuous function from X to Z. Continuous functions map compact (connected, path connected) sets into compact (connected, path connected) sets.
Further definitions to remember:

- Uniform convergence.
- Equicontinuous, uniformly continuous.
- Totally bounded.
- Compact, sequentially compact.
- Connected, path connected.
- Complete, completion.
- Banach space.
- Continuous linear mapping (=bounded) between vector spaces; operator norm.

Things to remember about compact sets:

- A is *compact* iff every sequence in A has a subsequence converging to a point in A;
- A closed subset of a compact set is compact;
- A continuous function attains a maximum and a minimum on a compact set (extreme value theorem);
- A continuous function on a compact set is uniformly continuous.
- For metric spaces, "compact" is equivalent to "totally bounded and complete" which in turn is equivalent to "sequentially compact;"
- Continuous image of a compact set is compact.

Things to remember about complete sets:

- A is *complete* iff every Cauchy sequence in A converges to a point in A;
- A closed subset of a complete metric space is complete;
- $\mathbf{R}^n$ is complete;

- the space of continuous functions on an interval where $\mathrm{dist}(f,g) = \max|f-g|$ is complete (here convergence is equivalent to the uniform convergence). The same is true for the space of bounded functions from an arbitrary metric space into a complete metric space (with the uniform distance); and for the space of continuous linear mappings from a normed vector space into a complete normed vector space (with the operator norm);
- The space l_p of sequences is complete for $1 \leq p \leq \infty$;
- The space $C([0,1])$ is *not* complete for d_p distance, $p < \infty$;
- every incomplete metric space possesses a completion.

Things to remember about convex, path connected and connected sets:

- a subset A of a vector space X is *convex* iff for every $x, y \in A$ the *segment* from x to y (i.e. the set $t \cdot x + (1-t) \cdot y, 1 \geq t \geq 0$) is contained in A;
- A is *path(wise) connected* iff every two points in A can be joined by a path (a continuous mapping from a closed interval into X) in A;
- A is *connected* iff it is not a union of two nonempty, disjoint, relatively open (in A) sets;
- a metric space X is connected if the only subsets of X which are both open and closed are the empty set and X itself;
- definitions of separation, connected components;
- under what conditions a union of connected sets is connected;
- a product of two connected sets is connected;
- (convex) implies (path connected) implies (connected), but not vice versa!
- a subset or $\mathbf{R}$ is connected iff it's an interval;
- a continuous function into $\mathbf{R}$ maps connected sets into intervals (i.e. intermediate value theorem holds for connected sets);
- a ball in $\mathbf{R}^n$ is convex.

Two more useful facts about such sets:

- an arbitrary intersection of convex sets is convex;
- an arbitrary union of (path) connected sets whose intersection is nonempty is also (path) connected.
- A uniform limit of continuous (integrable) functions is continuous (integrable).
- Limit of derivatives is equal to the derivative of the limit if the derivatives converge uniformly.
- Infinite series can be differentiated term by term in an interval inside the domain of convergence.

Theorems to know:

- Arzela-Ascoli.
- Dini's theorem.
- Baire's theorem.
- Stone-Weierstrass.
- Implicit and Inverse function theorems.
- Contraction Mapping theorem.
- Bernstein approximation theorem.

TO BE CONTINUED...