Notes on basic topology with examples from ℓ^p spaces, as pertaining to Analysis III

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1 Topology

This section is meant for those not taking geometry and topology 1, which spends a fair bit of time on point-set topology. We will do everything in the context of metric spaces, where things are a little easier, but note that there are much more general definitions. What is below simply serves as reference for later exercises.

Loosely speaking, the topology of a space is the structure that exists between the simple set and its geometry. For instance, the real numbers $\mathbb{R}$ are just a set, and once we have a notion of distance such as $d(x, y) = |x - y|$, we have a geometry – but there is something in between the two, which in particular measures notions of closeness and describes how things are "glued together." For the purposes of the course (and the midterm), pretty much all you need to know is: a topology on a space is just the collection of its open sets.

Definition. i) interior. The interior of a set A, denoted A° or $\text{int}(A)$ is the largest open set contained in A. That is, if V is open and $V \subset A$, then $V \subset A^\circ$, and A is open if it equals its own interior. In a metric space (X, d),

$$A^\circ = \{x \in X \ ; \ \exists\, \varepsilon > 0 \ : \ D(x, \varepsilon) \subset A\}.$$

In other words, A is open if $A = A^\circ$, which means that for any $x \in A$, there is an open set (in this case $D(x, \varepsilon)$) which contains x and fits inside A.

ii) closure. The closure of a set A, denoted $\overline{A}$ or $\text{cl}(A)$ is the smallest closed set containing A. That is, if K is closed and $A \subset K$, then $\overline{A} \subset K$. As for interior, A is closed if it equals its own closure. In a metric space (X, d),

$$\overline{A} = \{x \in X \ ; \ \forall\, \varepsilon > 0, \ D(x, \varepsilon) \cap A \neq \emptyset.\}$$

In other words, the closure of A are all the elements of x which are infinitesimally close to A.

iii) boundary. The boundary of a set A, denoted ∂A or $\text{bd}(A)$, is the intersection of the closure of A and its compliment, i.e.

$$\partial A = \overline{A} \cap \overline{A^c}.$$

In a metric space (X, d),

$$\partial A = \{x \in X \ ; \ \forall\, \varepsilon > 0, \ D(x, \varepsilon) \cap A \neq \emptyset \text{ and } D(x, \varepsilon) \cap A^c \neq \emptyset\}.$$

Remark. Note that a set A is open iff its complement is closed and A is closed iff its complement is open. The sets X and $\emptyset$ are both open and closed.

Example. The rationals $\mathbb{Q}$ are neither open nor closed. The negation of the statement of $\mathbb{Q}$ being open is: $\exists q \in \mathbb{Q} \ : \ \forall \varepsilon > 0, D(x,\varepsilon) \not\subset \mathbb{Q}$. Choose arbitrary $q \in \mathbb{Q}$ and let $\varepsilon > 0$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there is $\xi \in D(q,\varepsilon) \cap \mathbb{Q}^c$, i.e. there is an irrational number ξ in the ε ball of q. Since $\xi \notin \mathbb{Q}$, $D(q,\varepsilon) \not\subset \mathbb{Q}$ and so $\mathbb{Q}$ is not open. Similarly, $\mathbb{Q}$ is not closed; if it were, it would contain all elements $\mathbb{R}$ infinitesimally close to $\mathbb{Q}$, but it doesn't – take $\pi \in \mathbb{R}$, which satisfies the condition $\forall \varepsilon > 0$, $D(\pi,\varepsilon) \cap \mathbb{Q} \neq \emptyset$, yet is not in $\mathbb{Q}$. A better argument for $\mathbb{Q}$ not being closed is that its closure is the reals, so it is not its own closure and hence is not closed.

Remark. Note that int and cl do not "commute", i.e. $\overline{\mathbb{Q}^\circ} = \overline{\emptyset} = \emptyset$ while $\overline{\mathbb{Q}}^\circ = \overline{\mathbb{R}} = \mathbb{R}$.

Exercise. Prove that, in a metric space, $\overline{A} = \partial A \cup A^\circ$.

Definition. A topological space X is said to be **Hausdorff** if it seperates points, i.e. for any $x, y \in X$ with $x \neq y$, there are disjoint open sets U and V in the topology of X containing x and y, respectively, i.e. $x \in U$, $y \in V$ and $U \cap V = \emptyset$. An important property of Hausdorff spaces is that all one-point sets are closed (this is also known as a T_1 condition, which Dani Wise would say is a stupid name).

Exercise. Prove that any metric space (X, d) is Hausdorff.

Definition. A set K is said to be **compact** if any open cover of K contains a finite subcover. An open cover is an arbitrary collection of open sets (possibly uncountable). I've seen this definition misapplied many times; it *does not* mean that the set K has a finite open cover – *any* set has a finite open cover (the whole space X certainly covers K and is open by a previous remark). The next example might illustrate this better.

Example. The set $[0, 1]$ is compact, while $(0, 1)$ is not. Why? Take for instance the countable cover $\mathcal{C} = \{(1/n, \infty)\}_{n\in\mathbb{N}}$ for $(0, 1)$. This is an open cover, for any $x \in (0, 1)$ is in $\mathcal{C}$ for sufficiently large n – but we cannot do away with a finite number of sets of $\mathcal{C}$.

Remark. An important equivalent definition of compactness in a metric space is that every bounded sequence has a convergent subsequence. This is known as the Bolzano-Weierstrass theorem of real analysis (see, e.g., Bartle & Sherbert) and proves the sequential definition of compactness, i.e., every bounded sequence has a convergent subsequence.

Now we come to an important

Theorem 1 (Heine-Borel (sketch)). *In a finite-dimensional metric space, a set K is compact if and only if it is closed and bounded.*

Proof. That compact implies bounded is always true; compact implies closed when the space is Hausdorff. The converse is the most important statement, and we will use the sequential definition of compactness to prove it. Sequential compactness is equivalent to compactness when the topological space is actually a metric space (X, d). We sketch the idea of the proof, which utilizes the Bolzano-Weierstrass theorem.

Let $(x_n)^{(k)}$ be a bounded sequence in (X, d) a finite dimensional space, where $\dim X = d$ and $x_n \in \mathbb{R}$ for every $1 \leq n \leq d$ and $k \in \mathbb{N}$. Fix $n = 1$; the sequence $(x_1)^{(k)}$ is a bounded sequence of real numbers. By the Bolzano-Weierstrass theorem, there is a convergent subsequence $k_{1,i}$, $i \in \mathbb{N}$. Now, $(x_1)^{(k_{1,i})}$ is a convergent subsequence of real numbers, but $(x_n)^{(k_{1,i})}$ is still a subsequence of the original sequence, and $(x_2)^{(k_{1,i})}$ is also a sequence of real numbers which is bounded. By Bolzano-Weierstrass, there is a subsequence $k_{2,i}$ which is convergent. Repeat until we get to d. The sequence $(x_n)^{(k_{d,i})}$ converges for every n. We thus extract a convergent subsequence in n. $\square$

2 ℓ^p spaces

We will investigate these a good deal, as these are the first examples of infinite-dimensional spaces, and understanding them provides good intuition for the much more intricate L^p spaces (which will not be discussed here).

2.1 Basic properties.

(1) For $1 \le p \le \infty$, $\ell^p(\mathbb{N})$ is a vector space.

(2) For $1 \le p \le \infty$, $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$ is a norm on p. For $p = 2$, the parallelogram law holds, so $\ell^2(\mathbb{N})$ is a Hilbert space. For $0 < p < 1$, it is not a norm (the triangle inequality fails; take $x = (1, 0, \dots)$ and $z = (0, 1, 0, \dots)$).

(3) The triangle inequality for $1 \le p < \infty$ is the Minkowski inequality,

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

which in particular allows one to show that ℓ^p is a vector space (closure under addition follows from the triangle inequality).

(4) $\ell^1 \subset \ell^2 \subset \dots \subset \ell^\infty$. All absolutely convergent subsequences are square summable, but not all square summable sequences are absolutely convergent (e.g. $x_n = \frac{1}{n}$).

(5) The previous item makes it easy to see that $p \to \|\cdot\|_p$ is a decreasing function. To show this, write $\|x\|_p = e^{\frac{1}{p}\ln(\sum |x_n|^p)}$ and differentiate.

<u>Example</u>. This example illustrates that the $\|\cdot\|_\infty$ norm is not precise; elements that have a small distance from each other can be significantly different. Read this if you want an attempt[1] at a heuristic explanation.

Let (a_n) be a sequence of positive numbers, and consider the set

$$A = \{(x_n) \in \ell^\infty \mid |x_n| < a_n \ \ \forall \ n\}$$

It is an important exercise to show that in ℓ^p for $1 \le p < \infty$, A is open if and only if $\inf_n a_n > 0$, and a proof is included later on. If you don't remember the idea of the proof, you will not understand anything new by referring to it. Rather, first try to disprove the claim.

In the exercise, what allows $\inf\limits_n a_n > 0$ to imply that A is not open is the condition imposed by $p < \infty$. The crucial difference between $p < \infty$ and $p = \infty$ is the uniformity of the $p = \infty$ case. For every other p, *all* the terms matter; in the infinity case, *one* term matters – the rest have infinite freedom. The immediate consequence is that you've got ε room around all but one term. Conversely, in the $p < \infty$ case, we had to have $\lim\limits_{n\to\infty} |x_n| = 0$ for all $(x_n) \in \ell^p(\mathbb{N})$.

This is why A is never open in the $p = \infty$ case: even for $\inf\limits_n a_n > 0$, we can have $\inf\limits_n a_n - |x_n| = 0$ and $|x_n| < a_n$ (e.g. $x_n = a_n - \frac{1}{n}$). Then $(x_n) \in A$ but we do not have any

[1] It can only be attempt, as we're in infinite frikkin' dimensions, how do you want me to relate this to real life? What do WE know about infinity?

$\varepsilon > 0$ such that $D(x,\varepsilon) \subset A$. In the $p < \infty$ case, $\inf_n a_n - |x_n|$ is not possible as there are finitely many values further than some epsilon away from 0.

Note that this doesn't imply $\operatorname{int}(A) = \emptyset$, it just means the interior of A is a proper subset of A.

<u>Example</u>. On the other hand, elements of ℓ^∞ need not differ by much to have a lot of distance between them. While the previous example showed that the elements (which differ by $\frac{1}{2}$)

$$(\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{5}, 0, -\frac{2}{7}, \dots) \text{ and } (0,0,0,\dots)$$

can be somewhat 'close' together and still be very different, neither is it true that distance implies the elements *are* that different:

$$(0,0,0,1,0,\dots) \text{ and } (0,0,0,0,\dots)$$

This 'looseness' of ℓ^∞ allows us to create distance. An important consequence of this is that ℓ^∞ is not seperable; the elements are too far apart, or rather, it is too easy to make it appear so. We can exploit this by taking set of sequence with entries only 0 and 1, and we can create a distance of 1 between two sequences simply by making one entry differ by 1.

We first use the typical uncountability argument. If this set were countable, let $\{e^{(1)}, e^{(2)}, \dots\}$ be an enumeration. Consider the element $x = (x_n)$:

$$x_n = \begin{cases} 1 & \text{if } e_n^{(n)} = 0 \\ 0 & \text{if } e_n^{(n)} = 1 \end{cases}$$

Then (x_n) differs from every single term in the list, and so the set is uncountable.

Next comes abusing the coarseness of this norm. We can create uncountably many disjoint open sets, simply by taking open balls of radius $\frac{1}{2}$ around each element of this uncountable set. They each differ in at least one entry, so their distance is at least 1; therefore, the balls are disjoint, and so ℓ^∞ cannot be seperable.

Proposition 1. *Let (X,d) be a compact metric space. Then X is separable, i.e., there exists a countable dense subset.*

Proof. For each $n \in \mathbb{N}$, consider the collection of balls $\mathcal{C}_n = \{B(x, 1/n) : x \in X\}$; by compactness, there is a finite subcollection $B(x^{(n_1)}, 1/n), \dots, B(x^{(n_k)}, 1/n)$. Let $\mathcal{B}_n = \cup_{i=1}^{n_k} B(x^{(i)}, 1/n)$. $\mathcal{B}_n$ is a finite collection of open sets, so $\cup_{n\in\mathbb{N}}\mathcal{B}_n$ is a countable collection of open sets. Pick an x_j out of each open set in this collection. The set of all these x_j's is countable and dense. $\square$

Let now (a_n) be a sequence of positive numbers. Define:

$$A = \{(x_n) \in \ell^p \mid |x_n| < a_n \ \ \forall \ \ n\}$$
$$K = \{(x_n) \in \ell^p \mid |x_n| \le a_n \ \ \forall \ \ n\}.$$

Proposition 2. *For $1 \le p < \infty$, A is open if and only if $\inf_n a_n > 0$.*

Proof. ($\Rightarrow$) Suppose A is open, but that $\inf_n a_n \not> 0$. Since $a_n \ge 0$, we must have $\inf_n a_n = 0$. We have $x = 0 \in A$, so there is some $\varepsilon > 0$ such that $D(0,\varepsilon) \subset A$. Since the infimum of (a_n) is 0, there is $N \in \mathbb{N}$ such that $0 \le a_N < \frac{\varepsilon}{4}$. Define

$$y_n = \begin{cases} \frac{\varepsilon}{2} & n = N \\ 0 & \text{else} \end{cases}$$

Then $y \in D(0,\varepsilon)$, yet $y_N \geq a_N$, which is a contradiction. Hence A is not open.

($\Leftarrow$) Suppose $\delta = \inf_n a_n > 0$ and let $(x_n) \in A$ (such an (x_n) exists, as $0 \in A$). We need to get an ε-bound on all the x_n such that $|x_n \pm \varepsilon| < a_n$. Since $x \in \ell^p$, $\lim_{n\to\infty} x_n = 0$, so there is N such that $|x_n| < \frac{\delta}{2}$ for all $n \geq N$.

Next, let $\delta' = \min_{n \leq N} a_n - |x_n|$ (> 0 – why?). With this bound, for any $1 \leq n \leq N$, $|x_n \pm \delta'| < a_n$. Let now $\varepsilon = \min\{\delta, \delta'\}$, and take $y \in D(x,\varepsilon)$. Since

$$y \in D(x,\varepsilon) \Rightarrow \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} < \varepsilon \Rightarrow |x_n - y_n| < \varepsilon$$

we have for $1 \leq n \leq N$,

$$|y_n| \leq |y_n - x_n| + |x_n| < \varepsilon + |x_n| \leq a_n - |x_n| + |x_n| = a_n$$

while for $n \geq N$,

$$|y_n| \leq |y_n - x_n| + |x_n| < \varepsilon + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} < a_n$$

from which it follows that $|y_n| < a_n$ for all n, so $y \in A$ and $D(x,\varepsilon) \subset A$. Thus, A is open. $\square$

Remark. The set A is never open when $p = \infty$. To see this, let (a_n) be the relevant sequence of positive numbers, and suppose that A is open. Let $x_n = a_n - \frac{1}{n}$ for all n. Then $x_n < a_n$ for all n, yet there is no ε ball around (x_n).

Proposition 3. *The set K is compact in $\ell^\infty(\mathbb{N})$ if and only if $\lim_{n\to\infty} a_n = 0$.*

Proof. ($\Rightarrow$) This is the easy direction. Suppose that $\lim a_n \neq 0$, so that the limit is either strictly positive or does not exist. Let a_{n_k} be a subsequence such that $a_{n_k} > 0$ for each k, and let

$$x_i^{(n_k)} = \begin{cases} \frac{a_{n_k}}{2} & i = n_k \\ 0 & i \neq n_k \end{cases}.$$

Then $(x_i)^{(n_k)}$ has no convergent subsequence, so K is not compact.

($\Leftarrow$) Suppose that $\lim_{n\to\infty} a_n = 0$, and that (a_n) is a sequence of positive numbers in ℓ^∞. Let $(x_n)^{(k)}$ be a bounded sequence in ℓ^∞; we wish to extract a convergent subsequence. We will do so as in the proof of Heine-Borel, but we will need something extra to make the proof go through. In particular, we need to be able to show convergence of the sequence in ℓ^p. In finite dimensions, this could have looked something like

$$\|x - x^{(k)}\| \leq |x_1^{k_{1,i}} - x_1| + \cdots + |x_d^{k_{d,i}} - x_d| < \frac{\varepsilon}{n} + \cdots + \frac{\varepsilon}{n},$$

but we can't do that here as our ε/n would be 0. Instead, we must use some "control at infinity", which will come from the assumption that $\lim a_n = 0$. Use the diagonal argument as before to extract convergent subsequences for each $n \in \mathbb{N}$. [2] $\square$

[2] I should note that you guys have the Tychonoff theorem available to you, which states that an *arbitrary* product of compact sets is compact in the product (not box) topologies. This is fine too and I guess for your exam you can just state it; this is how you might prove it for countable dimensions.

3 Some more worked examples

EXAMPLE. Consider $(C([a,b]), \|\cdot\|_\infty)$, the space of continuous functions on a compact interval with the supremum norm. Let $F \in C([a,b])$ be fixed and $F(t) > 0$ for all t. Let $A = \{f \in C([a,b]) \ : \ |f(t)| < F(t) \ \forall \ t \in [a,b]\}$. Prove that A is open and describe $\overline{A}$, as well as ∂A. Let's write $X = C([a,b])$ for simplicity. To show that A is open, let $f \in X$, so that $f(t) < F(t)$ for all t. Consider the function $h(t) = F(t) - f(t)$, which is continuous. Since $[a,b]$ is compact, $h(t)$ achieves its maximum and minimum on $[a,b]$; let $\varepsilon = \min_{t\in[a,b]} h(t)$ and let $h(t_0) = \varepsilon$. Then, for any $g \in D(f, \varepsilon/2)$, we have

$$|g(t)| \leq |g(t) - f(t)| + |f(t)| < \frac{\varepsilon}{2} + |f(t)| \leq F(t) - f(t) + f(t) = F(t).$$

Thus, A is open.

Next, I claim that $K = \{f \in X \ : \ \forall t \in [a,b] \ \ , |f(t)| \leq F(t)\}$ is the closure of A. To show this, we must show that $\overline{A} \subset K$ and $K \subset \overline{A}$. As a general rule of thumb, $A \subset B \Rightarrow \overline{A} \subset \overline{B}$, and if B is closed $\overline{B} = B$. Clearly $A \subset B$, so it suffices to show B is closed and that $B \subset \overline{A}$.

To show that B is closed, we will show that B is open. Let then $g \in B^c$; we need to find an $\varepsilon > 0$ such that $D(g, \varepsilon) \subset B^c$. By definition of B^c, for all $t \in [a,b]$, $|g(t)| > F(t)$. Let $\varepsilon = \min_{t\in[a,b]} |g(t)| - F(t)$. For any $h \in D(g, \varepsilon)$, we have

$$|h(t)| \geq |g(t)| - |g(t) - h(t)| > |g(t)| - (|g(t)| - F(t)) = F(t).$$

Thus, B is closed. It remains to show that $B \subset \overline{A}$.

To show that any $g \in B$ is also in $\overline{A}$, we need to show that g satisfies the requirements of being in the closure (see the intro where closure is defined). That is, we need to show that for all $\varepsilon > 0$, $D(g, \varepsilon) \cap A \neq \emptyset$. It therefore suffices to exhibit a sequence of functions $f_n \in A$ which converge to g, so that g is a limit point of A and is hence in the closure.

We can pretty much take anytihng we want, like $f_n = (1 - 1/n)g$. Since $|g(t)| \leq F(t)$ for all t, we have $|f_n(t)| < F(t)$ for all t, so $f_n \in A$. It should be clear that $f_n \to g$, so $g \in \overline{A}$ and we are done.

From the relationship $\overline{A} = A^\circ \cup \partial A$, we have that

$$\partial A = \{f \in X \ : \ |f(t)| \leq F(t), \ \ \exists \, t_0 \in [a,b] : f(t_0) = F(t_0)\}.$$

EXAMPLE. Let $K(x,y)$ be continuous on $[a,b]^2$, $g(x)$ continuous on $[a,b]$. Show that, for all $\lambda \in \mathbb{R}$, there is a unique $f(x)$ such that

$$f(x) = \lambda \int_a^x K(x,y) f(y) \mathrm{d}y + g(x).$$

Define $A(f(x)) = \lambda \int_a^x K(x,y) f(y) \mathrm{d}y + g(x)$.

The obvious thing to try for is to use the Banach fixed point theorem, which will give you existence and uniqueness in one go. Of course, A need not be a contraction map – but if A^n is, then A^n has a unique fixed point, which will be the same as for A. To see this, suppose $A^2(f) = f$. Then

$$A(f) = A(A^2(f)) = A^2(A(f)),$$

so by uniqueness of the fixed point $A(f) = f$. By induction, $A^n(f) = f$.

First, let $M = \max_{[a,b]} |K(x,y)|$. Observe that

$$\begin{aligned} |Af(x) - Ah(x)| &= \left| \lambda \int_a^x K(x,y)[f(y) - h(y)] \mathrm{d}y \right| \\ &\leq |\lambda| \int_a^x |K(x,y)| f(y) - h(y)| \mathrm{d}y \\ &\leq M|\lambda| \|f - h\|_\infty (x - a). \end{aligned}$$

Thus, for sufficiently close f and h, we have $|Af(x) - Ah(x)| < \varepsilon$, so we have continuity for $A : C([a, b]) \to C([a, b])$. Now we want to find an n such that the map A^n is a contraction. Suppose for an induction hypothesis that

$$|A^{n-1}f(x) - A^{n-1}h(x)| \leq \lambda^{n-1}M^{n-1}\frac{(x-a)^{n-1}}{(n-1)!}\|f-h\|_\infty.$$

Then,

$$\begin{aligned} |A^n f(x) - A^n h(x)| &\leq M|\lambda|\|f-h\|_\infty \int_a^x \frac{(x-a)^{n-1}}{(n-1)^n}\mathrm{d}y \\ &= M|\lambda|\|f-h\|_\infty \frac{(x-a)^n}{n!}. \end{aligned}$$

Then, for any $x \in [a, b]$ (wlog $x = b$), there is n sufficiently large so that $d(Af, Ag)_\infty < 1$. By the Banach fixed point theorem, A^n has a fixed point, so by the previous argument so does A.

<u>Example</u>. Let k be a positive constant and $X \subset C1([a, b])$ be the set of all f such that

$$\int_a^b [f(t)^2 + f'(t)^2]\mathrm{d}t < k.$$

Prove that X is bounded and equicontinuous. By the fundamental theorem of calculus,

$$\begin{aligned} |f(x) - f(y)| &= \left|\int_y^x f'(t)\mathrm{d}t\right| \\ &\leq \left(\int_y^x \mathrm{d}t\right)^{1/2}\left(\int_y^x f'(t)^2\mathrm{d}t\right)^{1/2} \\ &= \sqrt{x-y}\left(\int_y^x f'(t)^2\right)^{1/2} \\ &\leq \sqrt{x-y}\left(\int_y^x f'(t)^2 + f(t)^2\mathrm{d}t\right)^{1/2} \\ &\leq \sqrt{k}\sqrt{x-y}. \end{aligned}$$

This shows equicontinuity.

The next example is important, and people get it wrong (almost always).

Proposition 4. *Completeness of ℓ^p for $1 \leq p \leq \infty$, $\ell^p(\mathbb{N})$ is Banach.*

Proof. Completeness means that every Cauchy sequence converges to an element in the space. Let $(x_n)^{(k)}$ be a Cauchy sequence in ℓ^p, $p < \infty$. We need to do two things:

i) Choose a candidate for $(x_n)^{(k)}$ to converge to, and

ii) Show that $(x_n)^{(k)}$ actually converges to this candidate.

The candidate will usually be obvious, as it is in this case – it should just be the pointwise limit[3]: $x_n := \lim_{k\to\infty} x_n^{(k)}$ for each n. We have to justify this. We will show that $x_n^{(k)}$ is Cauchy

[3] I will write (x_n) for the element in ℓ^p and x_n for the nth term of the element x. Thus, $(x_n)^{(k)}$ is an element in ℓ^p, and $x_n^{(k)}$ is the nth term of the kth element of the sequence.

for each n. Let $\varepsilon > 0$; there is $K \in \mathbb{N}$ such that $\|(x_n)^{(k)} - (x_n)^{(m)}\|_p < \varepsilon$ for all $k, m \geq K$. Thus,

$$|x_n^{(k)} - x_n^{(m)}| \leq \left(|x_n^{(k)} - x_n^{(m)}|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(m)}|^p\right)^{1/p} < \varepsilon.$$

Thus $x_n^{(k)}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, $\lim_{k\to\infty} x_n^{(k)} = x_n$ *exists*, and this, for *every* $n \in \mathbb{N}$.

Okay, so this is our candidate – is it in our space, ℓ^p? Let $\varepsilon = 1$. There is K such that $\|(x_n)^{(k)} - (x_n)^{(m)}\|_p < 1$ for all $k, m \geq K$. Hence,

$$\|(x_n)^{(k)}\| \leq \|(x_n)^{(k)} - (x_n)^{(m)}\|_p + \|(x_n)^{(m)}\|_p < 1 + \|(x_n)^{(m)}\|.$$

Since $(x_n)^{(m)} \in \ell^p$, $\|(x_n)^{(m)}\|_p < \infty$, so this is a finite upper bound which holds for every $k \in \mathbb{N}$. Call this last upper bound M.

Now, you *cannot* just take $k \to \infty$ here. Instead, we know that for every k, and fixed $N \in \mathbb{N}$,

$$\left(\sum_{n=0}^{N} |x_n^{(k)}|^p\right)^{1/p} \leq \|(x_n)^{(k)}\| < M.$$

We know that for each n, $\lim\limits_{k\to\infty} x_n^{(k)}$ exists. Since the sum is finite, we can pass the limit inside to obtain

$$\left(\sum_{n=0}^{N} |x_n|^p\right)^{1/p} \leq M.$$

This is true for *every* $N \in \mathbb{N}$, so *now* we can take $N \to \infty$ to obtain that $(x_n) \in \ell^p$.

Now we need to show that $(x_n)^{(k)}$ actually converges to (x_n) in ℓ^p. Let then $\varepsilon > 0$, and choose K such that $\|(x_n)^{(k)} - (x_n)^{(m)}\| < \varepsilon/2$. For any finite N,

$$\left(\sum_{n=1}^{N} |x_n^{(k)} - x_n^{(m)}|^p\right)^{1/p} < \frac{\varepsilon}{2}.$$

Fix $m \geq K$, and take $k \to \infty$. We can pass the limit inside *because the sum is finite.* We obtain

$$\left(\sum_{n=1}^{N} |x_n - x_n^{(m)}|^p\right)^{1/p} \leq \frac{\varepsilon}{2}.$$

Hence, take $N \to \infty$ to obtain the result.

$\square$

Proposition 5. *The space ℓ^p is seperable for $1 \leq p < \infty$, but not for $p = \infty$.*

Proof. Exercise. $\square$