SPECIAL SETS & SPACES & THEIR PROPERTIES

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Abstract. Here we wish to provide an extensive list of general topological/metric spaces X, stating some properties and special cases of these spaces relevant to an intermediate level real analysis course. The reader should note that this is not everything that is needed for such a course, but that it is simply a guideline.

1. Spaces of Functions

1.1. $\mathbf{X = C^k([a, b])}$.

(i) When equipped with the $d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$ distance, we have that X is complete. Moreover, it is also Banach.

(ii) The subset of all bounded functions on $[a, b]$ is closed.

(iii) X is always separable.

(iv) When equipped with the $d_p(f, g) = (\int_a^b |f(x) - g(x)|^p dx)^{\frac{1}{p}}$ distance, X is not complete. However, its completion is the space of all Lebesgue p-integrable functions (cool, eh?)

(v) If $\mathcal{F}$ is a family of functions $f \in X$, then $\mathcal{F}$ is sequentially compact if and only if $\mathcal{F}$ is uniformly bounded and equicontinuous.

(vi) $f \in X$ are always convex.

2. Finite Dimensional Spaces

2.1. $\mathbf{X = \mathbb{R}^n}$.

(i) $A \subseteq X$ is totally bounded if and only if A is bounded.

(ii) All open and half open subsets of X are connected for $n = 1$.

(iii) All star like sets $B \subseteq X$ is connected.

(iv) Closed bounded subsets of X are compacta.

2.2. $\mathbf{X = \mathbb{Q}}$.

(i) X can be completed to the p-adic rationals.

3. Infinite Dimensional Spaces

3.1. $\mathbf{X = l_\infty}$.

(i) X is Banach and thus it is complete.

(ii) X is also complete when equipped with the $d_\infty(x, y) = \sup_k |x_k - y_k|$.

(iii) When equipped with the uniform norm X is not separable.

(iv) The subset of all sequences of 0's and 1's is not separable.

3.2. $\mathbf{X = l_p}$.

(i) X is Banach and thus it is complete.

(ii) $X^* = (l_p)^* = l_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and hence X is reflexive for $1 < p < \infty$.

(iii) $(l_1)^* = l_\infty$ and $(l_\infty)*$ is a horrible space where only sadness and anguish exist (it is complete though).

(iv) For all $p \geq 1$ (the infinite case is allowed), the unit ball in X is never compact.

3.3. $\mathbf{X = l_2}$.

(i) X is not totally bounded, although the set

$$A = \left\{ x : |x_j| \leq 2^{-j} \right\} \subset X$$

is totally bounded.

3.4. $\mathbf{X = c_0}$ - The Space of Sequences with Zero Limit.

(i) $(c_0)^* = l_1$.

4. Weird Spaces

4.1. X is not Hausdorff.

(i) X is not separable.

(ii) X is not metrizable.

4.2. $\mathbf{X = K}$ - The Cantor Set.

(i) X is uncountable.

(ii) X has measure zero.

5. Continuous Functions

(i) They map compact sets to compact sets.

(ii) They map connected sets to connected sets if they are onto.

(iii) They map path-connected sets to path-connected sets if they are onto.

(iv) They map convex sets to convex sets.

(v) The inverse image of an open or closed set is open or closed respectively.

(vi) They are automatically uniformly continuous if defined on compact sets.

6. Complete Spaces

(i) The conjugate space of any space is complete.

(ii) Let X be complete, then $A \subseteq X$ is complete $\iff$ A is closed.

(iii) X is complete $\iff$ any sequence of nested balls with radii tending to zero must have nonempty intersection.

(iv) Every incomplete metric space has a completion that is unique up to isometry.

7. Compact Sets

I. Let X be complete. A subset $A \subset X$ is compact $\iff A$ is totally bounded, that is, A can be covered by finitely many ϵ-balls.

II. Let A be compact in X, then A is a compactum if and only if A is closed.

III. A metric space X is a compactum $\iff X$ is totally bounded.

IV. Every compactum has a countable dense subset.

V. A closed subset of a compact set is compact.

VI. A subset $Y \subseteq X$ is compact in $X \iff \overline{Y}$ is a compactum.

VII. Compact sets attain their sup and inf.

VIII. The infinite product (with the product topology) of compact sets is also compact.

8. Connected Sets

I. The only subsets in a connected set that are both open and closed are the empty set and the set itself.

II. Convex $\implies$ Path Connected $\implies$ Connected.

III. The closure of a connected set is connected.

IV. For A open and closed in X and $C \subseteq X$, we have that $C \cap A \neq \emptyset \implies C \subseteq A$.

V. An arbitrary union of (non-pairwise-disjoint) connected sets is connected.

VI. If X, Y are connected, then so is $X \times Y$.

9. Convex Sets

(i) The closure of a convex set is convex.

(ii) An arbitrary intersection of convex sets is convex.

10. Linear Functionals

(i) For $A : X \to Y$ where X, Y are normed and linear, we have that A bounded $\iff A$ continuous $\iff A$ continuous at some point in X.

(ii) $X \subseteq (X^*)^*$ and X is isomorphic to a linear subspace of $(X^*)^*$.

11. Fun Facts That Are Hard To Categorize

(1) $\overline{A}$ is closed for any set A.

(2) The kernel of any function on a metric space X is closed.

(3) The space of all functions satisfying a Lipschitz condition is not closed.

(4) Separately continuous bilinear maps are jointly continuous.

(5) Metric spaces are Hausdorff.