Honours Analysis III

Math 354

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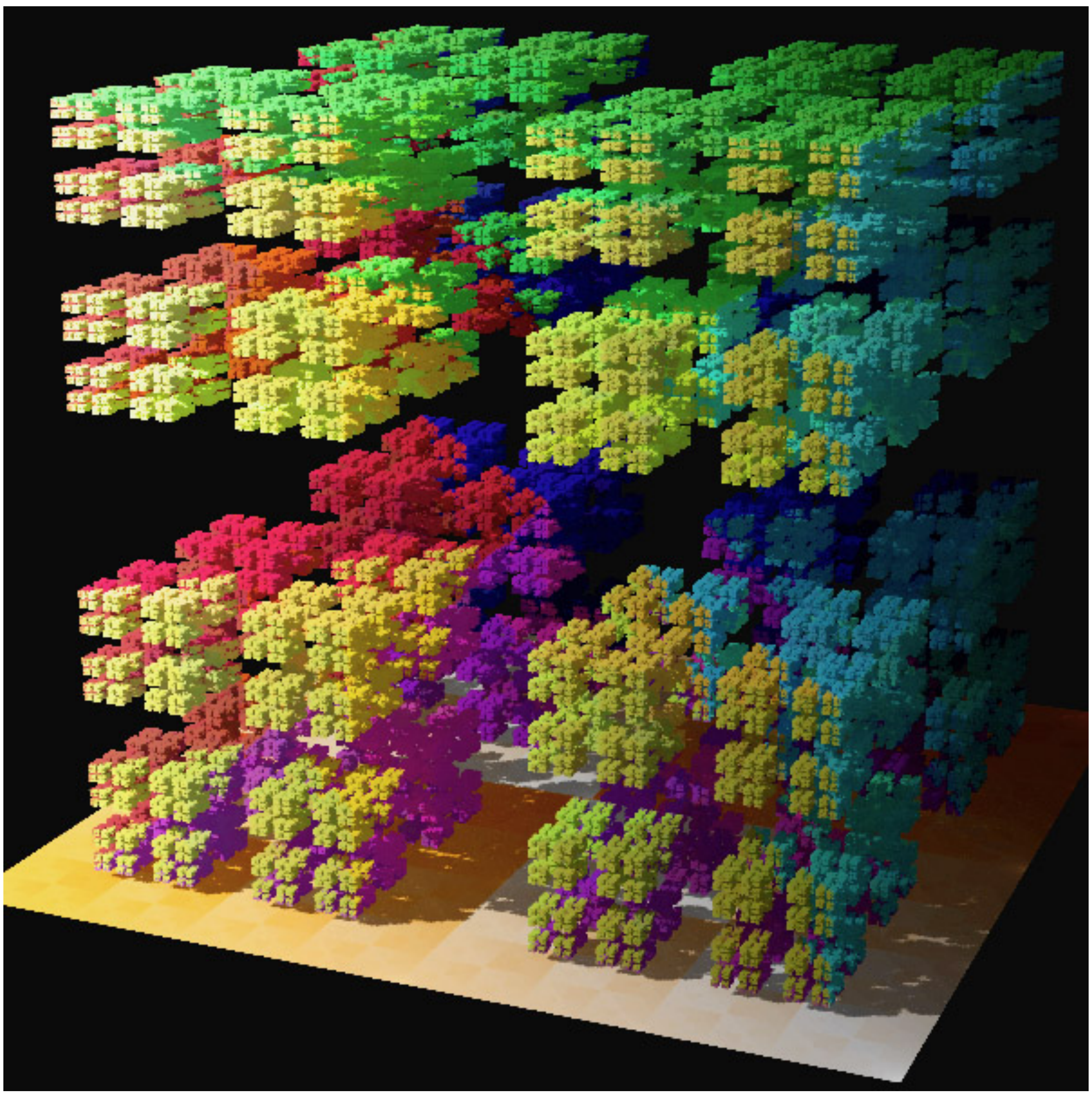

Contents

Overview

Definition 1. *Let X be a metric space. We define* ***Distance*** $d : X \times X \to \mathbb{R}$ *to satisfy*

(i) $\forall x \in X, \quad d(x, x) = 0$

(ii) $\forall x \neq y \in X, \quad d(x, y) > 0$

(iii) $\forall x, y \in X, \quad d(x, y) = d(y, x)$

(iv) $\forall x, y, z \in X, \quad d(x, z) + d(z, y) \geq d(x, y)$

Example 1. *The following are some of the main examples of metric spaces for this course.*

- *Spaces of sequences (finite or infinite) - with l_p norm.*
-
$$\{(x_1, \ldots, x_n), \ldots) : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

This leads us to a proposition.

Proposition 1. *If $\underline{x} = (x_1, x_2, \ldots), \underline{y} = (y_1, y_2, \ldots)$, then*

$$d(\underline{x}, \underline{y}) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p \right)^{1/p}$$

is a distance.

Example 2. *If $\underline{x} = \underline{0}$ and $\underline{y} = (1, 1/2, 1/3, 1/4, \ldots)$, then can we prove that*

$$d(\underline{x}, \underline{y}) = \sqrt{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{h^2} + \cdots} = \sqrt{\frac{\pi^2}{6}}$$

...probably not.

Example 3. *For $p \geq 1$, an example would be continuous functions f, g on $[a, b]$.*

Proposition 2. *In the space of continuous functions on $[a, b]$,*

$$d(f, g) = \left[\int_a^b |f(x) - g(x)|^p dx \right]^{1/p}$$

defines a distance. We recall the L_p norm to be

$$||f||_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

We now examine polygons in the plane. There is a problem in the assignment which asks if the difference of areas defines a distance. So we must prove that

$$d(P_1, P_2) = Area(P_1 \triangle P_2)$$

which is the area of

$$(P_1 \cup P_2) \backslash (P_1 \cap P_2)$$

Section 1.1

p-adic Distance

Let $p \in \{2, 3, 5, 7, 11, 13, \ldots\} = Primes$. Let $x, y \in \mathbb{Q}$ so

$$x = \frac{a_1}{b_1} \qquad y = \frac{a_2}{b_2}$$

Definition 2. *The p-**adic Norm** is defined as follows*

$$x = p^k \cdot \frac{c}{d}$$

where $k \in \mathbb{Z}$ and

$$||x||_p = p^{-k}$$

Definition 3. *The p-**adic Norm** is defined to be*

$$d_p(x, y) = ||x - y||_p$$

Example 4. *Working with p-adic distance. Suppose that*

$$x = \frac{24}{49}$$

then we have that

$$x = 7^{-2} \frac{24}{1} \Longrightarrow ||x||_7 = 7^2$$

$$x = 3^1 \frac{8}{49} \Longrightarrow ||x||_3 = 3^{-1}$$

$$x = 2^3 \frac{3}{49} \Longrightarrow ||x||_2 = 2^{-3}$$

Proposition 3. *$||x - y||_p$ defines a p-adic distance on $\mathbb{Q}$.*

NB: If x is as above, then

$$||x||_{Eucl} \cdot ||x||_2 \cdot ||x||_3 \cdot ||x||_7 = 1$$

Neat, eh?

Chapter 2

Introduction

Section 2.1

Normed Linear Spaces

Linear means exactly what you would think it means. A good way to show this is

$$(a_n)_{n=1}^{\infty} + (b_n)_{n=1}^{\infty} = (a_n + b_n)_{n=1}^{\infty}$$

$$(f+g)(x) = f(x) + g(x) \quad \forall\, x \in [a,b]$$

Definition 4. *Suppose that X is a linear space, then we say that the* ***Norm*** *of $x \in X$ is a map*

$$||\cdot|| : X \to \mathbb{R}_+$$

such that

(i) $||x|| = 0 \iff x = 0$

(ii) $||t \cdot x|| = |t| \cdot ||x||$

(iii) $||x+y|| \leq ||x|| + ||y||$

It is a fact that

$$d(x,y) = ||x-y||$$

defines a distance since

$$d(y,x) = ||y-x|| = ||-(x-y)|| = ||x-y|| = d(x,y)$$

$$d(x,z) = ||x-z|| = ||(x-y)+(y-z)|| \leq ||x-y|| + ||y-z|| = d(x,y) + d(y,z)$$

Section 2.2

Inner Product Spaces

An **Inner Product Space** is a space together with a map

$$(\cdot,\cdot) : X \times X \to \mathbb{R}$$

such that

(i) $0 \leq (x,x)$

(ii) $(x,y) = 0 \iff x = 0$

(iii) $(x, \alpha y + \beta z) = \alpha(x,y) + \beta(x,z), \quad x,y,z \in X, \alpha, \beta \in \mathbb{R}$

(iv) $(\gamma x + \delta y, z) = \gamma(x,z) + \delta(y,z)$

Example 5. *The following are some examples of inner products*

- *Dot Product*

$$(\underline{x}, \underline{y}) = \underline{x} \cdot \underline{y} = x_1 y_1 + \cdots + x_n y_n$$

- *Function Space Inner Product*

$$f, g \in C([a,b]) \Longrightarrow (f, g) = \int_a^b f(x)g(x)dx$$

and

$$||f|| = \sqrt{\int_a^b [f(x)]^2 dx}$$

PROPOSITION 4.

$$||x|| := \sqrt(x, x)$$

always defines a norm.
Proof. *We will need a lemma.*

LEMMA 5. *The following identity holds.*

$$(x, y) \le ||x|| \cdot ||y||$$

Proof.

$$(x + ty, x + ty) = (x, x) + 2t(x, y) + t^2(y, y)$$

we know that

$$||x + ty||^2 \ge 0$$

now,

$$D = 4(x, y)^2 - 4(x, x) \cdot (y, y) \le 0$$
$$(x, y) \le \sqrt{(x, x)} \cdot \sqrt{(y, y)} = ||x|| \cdot ||y||$$

With this lemma, we can now prove the above proposition. So
Proof.
The triangle inequality states that

$$||x + y|| \le ||x|| + ||y||$$

and now we can square both sides to obtain

$$(||x + y||)^2 \le (||x|| + ||y||)^2$$

and now

$$(x + y, x + y) = (x, x) + 2(x, y) + (y, y)$$

and

$$||x||^2 + 2||x|| \cdot ||y|| + ||y||^2 = (x, x) + 2||x|| \cdot ||y|| + (y, y)$$

by the lemma, and we cancel to obtain

$$\Longrightarrow (x, y) \le ||x|| \cdot ||y||$$

which yields our result. □
How to guess whether $||x||$ comes from (x, x) or not?
The answer is

$$(x + y, x + y) + (x - y, x - y) = (x, x) + 2(x, y) + (y, y) + (x, x) - 2(x, y) + (y, y)$$

then

$$(*) \qquad ||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$$

The norm comes from $(\cdot,\cdot) \iff (*)$ holds for all $x, y \in X$.

Proposition 6. *Let $(a_1, \ldots, a_n, \ldots)$ be such that*

$$\sum a_i^2 < \infty$$

and let $(b_1, \ldots, b_n, \ldots)$ be such that

$$\sum b_j^2 < \infty$$

and let

$$||\underline{a}|| = \left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2}$$

Then

$$\sum_{i=1}^{\infty} a_i b_i \leq ||a|| \cdot ||b||$$

and so

$$(\underline{a}, \underline{b}) = a_1 b_1 + \cdots + a_n b_n \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{j=1}^{n} b_j^2} \to ||a|| \cdot ||b||$$

as $n \to \infty$.
Proof.
To prove convergence,

$$\sum_{k=n}^{n+m} a_i b_i \leq \sqrt{\sum_{k=n} a_k^2} \cdot \sqrt{\sum_{k=n}^{n+m} b_k^2}$$

the two components on the right converge separately to 0 as $n \to \infty$. This is because

$$\sum_{k=1}^{\infty} a_k^2 < \infty$$

and the same goes for the sum of b_k^2. Thus, the whole thing converges to 0.
Summarizing, we proved that $(C([a,b]), L^2$-norm$)$ is a metric space, and so is l^2. □

Next, we look at L_p-norm which is

$$||f||_p = \left[\int_a^b |f(x)|^p dx\right]^{(1/p)}$$

and at l_p-space, $p \neq 2$, we have

$$||\underline{a}||_{l_p} = \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{(1/p)}$$

Thus, l_p is the space of all $\underline{a}$ such that

$$\sum_{k=1}^{\infty} |a_k|^p < \infty$$

To prove this we need a lemma.

LEMMA 7. *Let $a > 0, b > 0$, and $p, q \geq 1$ and such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

*NB: We say that p, q are **Conjugate Exponents**. Then*

$$(*) \qquad a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof.
Let $x > 0$, then $f(x) = \log(x)$ which is a concave function.

$$f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2} < 0$$

and since f is concave, we can say that

$$f(\alpha x + \beta y) \geq \alpha f(x) + \beta f(y)$$

where $\alpha + \beta = 1$. Also, we know that

$$\begin{aligned} \log(a \cdot b) &= \log a + \log b \\ &= \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) \\ (**) \qquad &\leq \log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \end{aligned}$$

THEOREM 8 (Holder's Inequality). *Let $p < 1$*

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1$$

Remark. *Both sides are homogeneous of degree 1 in $(a_k), (b_k)$. So WLOG, we can rescale $a_k - s$ so that*

$$\sum_{k=1}^{n} |a_k|^p = 1$$

and

$$\sum_{k=1}^{n} |b_k|^q = 1$$

so we say that

$$\begin{aligned} \sum_{k=1}^{n} |a_k b_k| &\leq \sum_{k=1}^{n}\left(\frac{|a_k|^p}{p} + \frac{|b_k|^q}{q}\right) \\ &= \frac{1}{p}\left(\sum_{k=1}^{n} |a_k|^p\right) + \frac{1}{q}\left(\sum_{k=1}^{n} |b_k|^q\right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \end{aligned}$$

by our normalization of the sums. Now,

$$RHS = \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}} = 1^{1/p}1^{1/q} = 1$$

and so

$$LHS \leq RHS$$

Now, if we let $n \to \infty$, then we get Holder for infinite series. The proof is left to the reader as an excercise.

THEOREM 9 (Minkowski Inequality). *(For Sequences With $n < \infty$)*
Recall that for $\underline{x} = (x_1, \ldots, x_n)$, we have

$$||\underline{x}||_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$$

We want

$$||\underline{x} + \underline{y}||_p \leq ||\underline{x}||_p + ||\underline{y}||_p$$

So, we know that

$$\begin{aligned}
||\underline{x} + \underline{y}||_p^p &= \sum_{k=1}^{n} |x_k + y_k|^p \\
&\leq \sum_{k=1}^{n} (|x_k| + |y_k|)^p \\
&= \sum_{k=1}^{n} [(|x_k| + |y_k|)^{p-1}|x|_k + (|x_k| + |y_k|)^{p-1}|y|_k] \\
&\text{and we apply Holder to each inner sum and let} \\
&b_k = (|x_k| + |y_k|), a_k = |x_k| \\
&\leq \left(\sum k = 1^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} [(|x_k| + |y_k|)^{p-1}]^q\right)^{1/q} \\
&\quad + \left(\sum k = 1^n |y_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} [(|x_k| + |y_k|)^{p-1}]^q\right)^{1/q}
\end{aligned}$$

And now,

$$\frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow 1 - \frac{1}{p} = \frac{1}{q} \Longrightarrow \frac{p-1}{p} = \frac{1}{q}$$
$$\Longrightarrow (p-1)q = p$$

and so

$$RHS = \left[\left(\sum k = 1^n |x_k|^p\right)^{1/p} + \left(\sum k = 1^n |y_k|^p\right)^{1/p}\right] \cdot \left[\sum k = 1^n (|x_k| + |y_k|)^q\right]^{1/q}$$

and

$$RHS \leq LHS = \sum_{k=1}^{n} (|x_k| + |y_k|)^p$$

$$\Longrightarrow \left[\sum_{k=1}^{n} (|x_k| + |y_k|)^p\right]^{1-1/q} \leq ||\underline{x}||_p + ||\underline{y}||_p$$

and we can conclude that

$$||\underline{x}+\underline{y}||_p \leq ||\underline{x}||_p + ||\underline{y}||_p \qquad \square$$

EXAMPLE 6. *The unit sphere in*

$$l_p(\mathbb{R}^n) = \left\{ \underline{x} \in \mathbb{R}^n : \sum_{k=1}^{n} |x_k|^p = 1 \right\}$$

For $n = 2$, $p = 1$, *we have*

$$\{(x, y) : |x| + |y| = 1\}$$

For $n = 2$, $p = 2$, *we have*

$$\{(x, y) : |x|^2 + |y|^2 = 1\}$$

Now we ask what happens when $p \to \infty$*? We get*

$$[|x_k|^p + |y_k|^p]^{1/p} = (|x_k|^p)^{1/p} \cdot \left(1 + \left(\frac{|y_k|}{|x_k|}\right)^p\right)^{1/p} \to x$$

as $p \to \infty$. *Now as another example, we have*

$$||(x, y)||_p \to \max\{|x|, |y|\} = ||(x, y)||_\infty$$

as $p \to \infty$.

DEFINITION 5. *The* l_∞ *norm of* $\underline{x} = (x_1, \ldots, x_n)$ *is*

$$||\underline{x}||_\infty = \max_{k=1}^{n}\{x_k\}$$

and for $f \in C([a, b])$, *we get that the* l_∞ *norm of* f *is*

$$||f||_\infty = \max_{x\in[a,b]} |f(x)|$$

and we can show that

$$||f||_p \to ||f||_\infty$$

as $p \to \infty$, *for any* $f \in C([a, b])$.

SECTION 2.3

METRIC SPACE TECHNIQUES

DEFINITION 6. *We say that* (X, d) *is a* ***Metric Space*** *if* $d(\cdot, \cdot)$ *defines a distance on the set* X.

DEFINITION 7. *Let* $A \subset X$ *where* X *is a metric space. Let* $x \in X$ *(may or may not be in* A*). If every ball* $B(x, r)$ *centred at* x *of radius* r *has at least one point from* A *for any* $r > 0$, *then this is equivalent to calling* x *a* ***Contact Point****. Also, just for notational purposes,*

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

Remark: Any $x \in A$ is a contact point of A.

DEFINITION 8. *If $B(x,r)$ for any $r > 0$ has infinitely many points from A, then we say that x is a* ***Limit Point****.*

DEFINITION 9. *If $\forall\, x \in A, \exists\, r > 0$ s.t.*

$$B(x,r) \cap A = \{x\}$$

PROPOSITION 10. *Let $x \in X$ be a contact point. Then, x must be one of the following.*

- *x is an isolated point $x \in A$*
- *x is a limit point of A, $x \in A$*
- *x is a limit point of A, $x \notin A$*

Also, x is not an isolated point if and only if $\exists (x_n)_{n=1}^{\infty} \in A$ where the x_n are distinct such that $x_n \to x$ as $n \to \infty$ and thus

$$\lim_{n\to\infty} d(x, x_n) = 0$$

DEFINITION 10. *The* ***Closure*** *$\overline{A}$ of A is the set of all contact points.*

$$\overline{A} = A \cup \{\text{Limit points of } A\}$$

PROPOSITION 11.

$$\overline{\overline{A}} = \overline{A}$$

Proof.
Let $x \in \overline{\overline{A}}$ and let $r > 0$. We know that $\exists\, y \in A$ such that

$$y \in B(x_1, r_1) \subseteq B(x,r)$$

So, x is a contact point of A, and thus $x \in \overline{A}$. $\square$

PROPOSITION 12. *(i) $A_1 \subseteq A_2 \Longrightarrow \overline{A_1} \subseteq \overline{A_2}$*
(ii) $A = A_1 \cup A_2 \Longrightarrow \overline{A} = \overline{A_1} \cup \overline{A_2}$

DEFINITION 11. *$A, B \subset X$. We say that A is* ***Dense*** *in B if*

$$B \subseteq \overline{A}$$

and it is also true that A is dense if and only if A is dense in X.

EXAMPLE 7. *Points with rational coordinates are dense in $\mathbb{R}^k$.*

DEFINITION 12. *We say that X is* ***Separable*** *if X has a countable, dense subset.*

EXAMPLE 8. *Let X be the set of all bounded sequences of real numbers. Distance on X can be the l_∞ distance.*

$$d(\underline{x}, \underline{y}) = \sup_n |x_n - y_n|$$

$$||(x_1, \ldots, x_n, \ldots)||_\infty = \sup_n |^n x_n|$$

$$\underline{x} = (0.9, 0.99, \ldots, 0.99\cdots 99, \ldots)$$

$$||\underline{x}||_\infty = \sup_n \left(\frac{9}{10} + \cdots + \frac{9}{10^n}\right) = 1$$

PROPOSITION 13. *X_∞ with l_∞ distance is not separable.*
Look at $A \subset X$, $A = \{$ all infinite sequences of 0's and 1's$\}$. A is not countable by the Cantor Diagonalization Argument.
Proof.
Suppose it is

$$\underline{a_1} = (\epsilon_1^1, \ldots \epsilon_1^k, \ldots)$$

$$\underline{a_2} = (\epsilon_2^1, \ldots \epsilon_2^k, \ldots)$$

$$\underline{a_3} = (\epsilon_3^1, \ldots \epsilon_3^k, \ldots)$$

$$\underline{b} = (b_1, \ldots, b_n, \ldots)$$

If $\epsilon_1^1 = 1$ then $b_1 = 0$. If $\epsilon_1^1 = 0$ then $b_1 = 1$. If $\epsilon_2^2 = 0$, then $b_2 = 1$. If $\epsilon_2^2 = 1$, then $b_2 = 0$.
Claim: *Sequence $\underline{b}$ is different from all a_i. This is a contradiction which shows that A is uncountable. A has cardinality of the continuum.*
Claim: *If $\underline{x_1}, \underline{x_2} \in A$, then $d_\infty(\underline{x}_1, \underline{x}_2) = 1$.*

LEMMA 14. *A is not separable.*
Proof.

$$\forall\, x \in A, \exists\, y \in B \ s.t.\ y \in B(x, 1/3)$$

Suppose that B is a countable dense subset of A.
Claim: *If $x_1 \neq x_2, \in A$, $y_1, y_2 \in B(x_1, 1/3)$, then $y_1 \neq y_2$. This is a contradiction!* □

Let $A \subseteq X$ be uncountable. Let $x, y \in A, x \neq y$ and $d(x, y) \geq 1$.
We claim that if $B \subset X$ is dense in X, then B cannot be countable.
Let $(x_\alpha)_{\alpha \in A}$ be our set. Consider

$$\{B(x, 1/3) : x \in A\}$$

The set B is dense in X, so $\forall \alpha \in A$, $B(x_\alpha, 1/3)$ contains a point $y_\alpha \in B$.
LEMMA 15. *If $x_\alpha \neq x_\beta$, then $B(x_\alpha, 1/3) \bigcap B(x_\beta, 1/3) = \emptyset$.*
Proof.
We know that $y_\alpha \in B(x_\alpha, 1/3)$ and $y_\beta \in B(x_\beta, 1/3)$. Moreover, $y_\alpha \neq y_\beta$.
It follows that there is a bijection between A and a subset of B. □

Examples Of Countable Dense Sets In $C([a, b])$ With Various Distances
Suppose that $f \in C^k([a, b])$. The first idea is to approximate by polynomials. Bernstein polynomials approximate well.

$$P_n(x) = a_n x^n + \cdots + a_0$$

where $a_j \in \mathbb{R}$ and

$$Q_n(x) = b_n x^n + \cdots + b_0$$

where $b_j \in \mathbb{Q}$. Those polynomials are dense in $C([a, b])$ with both L_p and L_∞.

$$d_p(f, g) = \left[\int_a^b |f(x) - g(x)|^p dx\right]^{1/p}$$

$$d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

$$\sum_{k=-N}^{N} a_k \sin(kx) + b_k \cos(kx)$$

Definition 13. *The* ***distance from a point*** *x* ***to a set*** *A is equivalent to*

$$d(x, A) = \inf_{a \in A} d(x, a)$$

For example, $d(x, A) = 0$ if and only if x is a contact point of A.

$$d(A, B) = \inf_{x \in A, y \in B} \{d(x, y)\}$$

Another example is that if $A \cap B \neq 0$, then we can take $x = y \in A \cap B$. Hence $d(A, B) = 0$. It's not true both ways. We can have $A \cap B \neq 0$ but $d(A, B) = 0$.

Definition 14. *Let $A \subset X$. We say that A is* ***Closed*** *if $\overline{A} = A$.*

Example 9. *Some examples of closed sets are*

- $[a, b] \subset \mathbb{R}$
- $\{y \in X : d(y, x_0) \leq R > 0\}$ *or* $\{f \in C([a, b]) : |f(x)| \leq R\}$ *for all* $x \in [a, b]$.

If $f(x) = 0$ on $[a, b]$, then

$$d(0, g) = \sup_{x \in [a,b]} |g(x)|$$

and

$$\{g : d(0, g) \leq R\}$$

which is exactly the second item on the list above.

Proposition 16. *Let $(A_\alpha)_{\alpha \in I}$ be a collection of closed sets. Then*

$$B = \bigcap_{\alpha \in I} A_\alpha$$

is also closed.

Proof.

$$B = \bigcap_{\alpha} A_\alpha \Longrightarrow \overline{B} = B \bigcup \{\textit{Limit Points of } B\}$$

And $\overline{B} = B \iff$ every limit point x of B belongs to B. So, suppose that x is a limit point of $B = \cap_\alpha A_\alpha$. Let $r > 0$, then $B(x, r)$ contains infinitely many points of

$$B = \bigcap_{\alpha} A_\alpha$$

If $y \in B$, then $y \in A_\alpha$ for all α. This means that x is a limit point of A_α for all $\alpha \in I$. A_α is closed, and so every limit point is contained in A_α. Thus, $x \in A_\alpha$ for all $\alpha \in I$.

$$\Longrightarrow x \in \bigcap_{\alpha \in I} A_\alpha = B \qquad \square$$

Proposition 17. *Let $A_1, \ldots, A_n$ be closed. Then*

$$B = \bigcup_{i=1}^{n} A_i$$

is closed.
Proof.

$$B = \bigcup_{i=1}^{n} A_i$$

Let $x \in B$. We shall show that x cannot be a limit point of B. if

$$x \notin \bigcup_{i=1}^{n} A_i$$

then $x \notin A_i$ for all $i = \{1, \ldots, n\}$. All the A_i are closed, so x cannot be a limit point of A_i for any i. This implies that $\exists\, r > 0$ such that $x \notin A$. Let

$$r = \min_k d(x, y_k)$$

then

$$B(x, r) \bigcap A = \emptyset$$

for any $1 \leq i \leq n$. Now $\exists\, r_i$ such that

$$B(x, r_i) \bigcap A_i = \emptyset$$

Let

$$r = \min_{1 \leq i \leq n} r_i$$

then

$$B(x, r) \bigcap A_i = \emptyset$$

for each i. Thus, x is not a limit point of

$$\bigcup_{i=1}^{n} A_i \qquad \square$$

Definition 15. *We say that x is an **Interior Point** of A if and only if*

$$\exists\, r > 0 \text{ s.t. } \quad B(x, r) \subset A$$

Definition 16. *We say that A is **Open** if every point in A is an interior point.*

Proposition 18. *A is open if and only if $X \backslash A$ is closed.*
Proof.
Suppose that A is open, and that $x \in A$, then $B(x, r) \subset A$ and

$$B(x, r) \bigcap A^C = \emptyset$$

Hence, x is not a contact point of A^C. So, $(A^C) \subseteq \overline{A^C} \Longrightarrow A^C$ is closed.
If A^C is closed, then $x \in A \Longrightarrow x$ is not a contact point of A^C. This means that $\exists\, r > 0$ such that

$$B(x, r) \bigcap A^C = \emptyset$$

$$\Longrightarrow B(x, r) \subset A$$

$\Longrightarrow x$ is an interior point of A. $\square$

Proposition 19. *If A_α is open for any $\alpha \in I$, then*

$$B = \bigcup_{\alpha \in I} A_\alpha$$

is also open.
Proof.
A_α is open and so equivalently, we have that A_α^C is closed

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^C = \bigcap_{\alpha \in I} A_\alpha^C$$

is closed. So, B^C is closed which is equivalent to B being open. Now we invoke Proposition 17. $\square$

Proposition 20. *A finite intersection of open sets is also open. That is, if A_i is open for $i = 0, \ldots, n$, then*

$$\bigcap_{i=1}^{n} A_i$$

is also open. Beware, however, that this is not necessarily true for the infinite case. An example of this would be that if

$$A_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) \subset \mathbb{R}$$

which is clearly open for any n, then

$$B = \bigcap_{n=1}^{\infty} A_n = [0, 1]$$

which is closed.

Definition 17. *A collection A_α, for $\alpha \in I$ of open sets is called a* ***Basis*** *(of all open sets) if and only if any open set in X is a union of a sub-collection of A_α.*

Definition 18. *X is called* ***Second Countable*** *if and only if there is a countable basis of open sets of X.*

Lemma 21. *$\{G_\alpha\}$ forms a basis if and only if for any open set A, and for any $x \in A$, there exists α such that*

$$x \in G_\alpha \subset A$$

Proposition 22. *Let X be a metric space. We claim that X is second countable if and only if X is separable.*
Proof.
Idea is to let $\{y_1, y_1, \ldots, y_n, \ldots\}$ be countable, dense subset of X. Then,

$$\{B(y_j, r_j) : r_j \in \mathbb{Q}\}$$

is a basis of all open sets in X.
$(\Longrightarrow)$ *Let $G_1, \ldots, G_n$ be a countable basis. Choose $x_n \in G_n$. We claim that the set $\{x_n\}$ is dense in X. To prove this, we let $x \in X$, $r > 0$ and we consider $B(x, r)$ which is an open set. Now, by the lemma above, we know that there exists m such that $x \in G_m \subset B(x, r)$. It follows that $x_m \in G_m \subset B(x, r)$.*
$(\Longleftarrow)$ *Let $\{x_n\}$ be a countable dense subset of X. It suffices to show that*

$$\left\{B\left(x_n, \frac{1}{k}\right) : n, k \in \mathbb{N}\right\}$$

forms a countable basis. To show this , we let A be an open subset of X and we pick $x \in A$. Choose $m > 0$, such that

$$B\left(x, \frac{1}{m}\right) \subset A$$

Next, we choose k such that

$$d(x, x_k) < \frac{1}{3m}$$

We now claim that

$$x \in B\left(x_k, \frac{1}{2m}\right) \subset B\left(x, \frac{1}{m}\right) \subset A$$

$$\frac{1}{2m} + \frac{1}{3m} = \frac{5}{6m} < \frac{1}{m}$$

This claim implies the previous claim by the lemma. $\square$

Fact: $\emptyset, X$ are both open and closed. By X, we mean the entire metric space.

DEFINITION 19. *We say that X is* ***Connected*** *if and only if any subset $A \subset X$ that is both open and closed is either $\emptyset$ or X.*

EXAMPLE 10. *$\mathbb{R}$ is connected, but $\mathbb{R} - \{0\}$ is not.*

DEFINITION 20. *Let d_1, d_2 be two distances on X. We say that d_1 and d_2 are* ***Equivalent*** *precisely if there are two constants $0 < c_1 < c_2 < \infty$ such that*

$$c_1 < \frac{d_1(x, y)}{d_2(x, y)} < c_2$$

This implies that for $r_2 < r_1 < r_3$, we have

$$B_{d_2}(x, r_2) \subset B_{d_1}(x, r_1) \subset B_{d_2}(x, r_3)$$

EXCERCISE: Express r_1, r_3 if we know c_1, c_2.
Finally, if d_1, d_2 are equivalent, then they define the same open and closed sets.
These constants are

$$r_1 = r, r_2 = \frac{r}{C}, r_3 = r \cdot C$$

and

$$c_1 = \frac{1}{C}, c_2 = C$$

for $C > 1$.
COROLLARY 23. *Open sets with respect to d_1, d_2 are the same.*

COROLLARY 24. *The topologies (space X together with a collection of open sets) defined by d_1, d_2 are the same.*

DEFINITION 21. *Let X be a set and let $\{A_\alpha\}_{\alpha \in I}$ be a collection of open sets. A* ***Topological Space*** *satisfies*

(i) $\emptyset, X$ are both open.

(ii)

$$\bigcup_{\alpha \in J} A_\alpha$$

is open.

(iii)

$$\bigcap_{i=1}^{n} A_i$$

is open.

PROPOSITION 25. *Distances in $\mathbb{R}^n$ defined by*

$$d_p(\underline{x}, \underline{y}) = \left(\sum_{j=1}^{n} |x_i - y_i|^p \right)^{1/p}$$

for $p \geq 1$

$$d_\infty(\underline{x}, \underline{y}) = \max_{1 \leq j \leq n} |x_i - y_i|$$

are equivalent.

DEFINITION 22 (More General Definition). *We say that x is a* ***Contact Point*** *of $A \subset X$ where X is a topological space if every neighbourhood of x contains a point in A.*

A sequence $(x_n)_{n=1}^{\infty} \to x$ if and only if for any neighbourhood U of x, there exists $N > 0$ such that $x_n \in U$ for $n \geq N$.
Metric Spaces are Hausdorfff.
DEFINITION 23. *We say that a space X is* ***Hausdorff*** *if for any $x, y \in X$ such that $x \neq y$, there exists r_1, r_2 such that*

$$B(x, r_1) \cap B(y, r_2) = \emptyset$$

OR
For any $x \neq y$, there exists open sets U containing x and V containing y such that

$$U \cap V = \emptyset$$

DEFINITION 24. *We say that a topological space X is* ***Metrizable*** *if there exists a metric d on X such that open sets defined by d give the same topology on X.*

COROLLARY 26. *If a topological space is not Hausdorff, then it is not metrizable.*

DEFINITION 25. *Let $f : X \to Y$. We say that f is* ***Continuous*** *at $x \in X$, if*

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t. \ \forall y \in X, d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

and we say that a f is a ***Continuous Function*** *if it is continuous $\forall x \in X$.*
OR

$$\forall (x_n)_{n=1}^{\infty} \to x \ \ (\lim_{n\to\infty} x_n = x), \lim_{n\to\infty} f(x_n) = f(x)$$

PROPOSITION 27. *$f : X \to Y$ is continuous at every point $x \in X$ if and only if for every open set U in Y,*

$$f^{-1}(U) \subset X$$

is also open. Moreover, this is also equivalent to saying that for every closed set B in Y,

$$f^{-1}(B) \subset X$$

is also closed. Another way to say this is

$$X - f^{-1}(B) = f^{-1}(Y - B)$$

Proof.
$(\Longrightarrow)$ *Let $B \subset Y$ be closed and let $f : X \to Y$ be continuous. Now let $f^{-1}(B) = A \subseteq X$. We want A to be closed. It suffices to show that all limit points of A lie in A. We know that there exists $(x_n) \in A$ such that $x_n \to x$ as $n \to \infty$, and that f is continuous at x. By the definition of continuity, we have that*

$$f(x_n) \to f(x)$$

Now, $f(x_n) \in B$, and B is closed. Thus $f(x) \in B$, but then $x \in f^{-1}(B) = A$. $\square$ $(\Longleftarrow)$ *Let $x \in X$, $y = f(x)$ and let U be an open set, and $y \in U$. Then $Y - U$ is closed. Therefore, $f^{-1}(Y - U) = f^{-1}(A)$ is closed. Moreover, $x \notin A$. So there exists an open set V where $x \in V \subset X - A$. Therefore, $f(V) \subset U$. For any sequence $x_n \to x$, $x_n \in V$ for $n \geq N$. Then $f(x_n) \in U$, and so*

$$f(x_n) \to f(x) \quad as \quad n \to \infty$$

Claim: Let K be the Cantor set. We claim that K is uncountable.
The idea for the proof is that points in K are like real numbers with only 0's and 2's in expansion (base 3). If $x \in [0, 1]$, then for a Decimal Expansion, we have

$$x = \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} + \cdots$$

for $a_j \in \{0, 1 \ldots, 9\}$. We can also use base 2 for a Binary Expansion

$$x = \frac{a_1}{2} + \cdots + \frac{a_n}{2^n} + \cdots$$

for $a_j \in \{0, 1\}$. For the Cantor set, we would use a Ternary Expansion

$$x = \frac{a_1}{3} + \cdots + \frac{a_n}{3^n} + \cdots$$

for $a_j \in \{0, 1, 2\}$.
Proposition 28. *Points in K are in one to one correspondence with*

$$\sum_{j=1}^{\infty} \frac{a_j}{3^j}$$

such that $a_j \in \{0, 2\}$.
Proof.
The cantor set K is self similar. To show this we define a map

$$f : K \bigcap \left[0, \frac{1}{3}\right] \to K$$

where $f(x) = 3x$. If we look at a ternary expansion in "decimal notation" (i.e. $0.22222 \approx 0.100000$), then multiplying the ternary expansion of a number is just the same as multiplying a decimal expansion by 10. $\square$

We now examine how to prove that a set $A \in X$ is dense in $B \in X$. Assume for simplicity that $A \subset B$. We need to show that

$$\forall x \in B, \forall \epsilon > 0, \quad \exists\, y \in B(x, \epsilon) \cap A$$

Now, if we construct the cantor set by letting $[0,1] = I_1^0$ and letting the following two intervals to be I_1^1, I_2^1 where the superscript means the step and the subscript means the enumeration of the interval within the set. Thus, at step n, there are 2^n intervals written as

$$I_1^n, I_2^n, \ldots, I_{2^n}^n$$

each of which has length

$$|I_j^n| = \frac{1}{3^n}$$

Now, every $x \in K$ can be written as

$$x = \bigcap_{k=1}^{\infty} I_{m_k}^k$$

We also make the claim that K has length 0. The sum of the intervals that we remove is

$$\frac{1}{3} + 2\frac{1}{9} + 4\frac{1}{3^3} + \cdots + \frac{2^n}{3^{n+1}} + \cdots$$

which is a geometric progression

$$\frac{1}{3}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i = \frac{1/3}{1-2/3} = 1 = |[0,1]|$$

And thus, this set has length 0.

Theorem 29. *If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then*

$$g \circ f : X \to Z$$

is also continuous
Proof.
Let $U \subset Z$ to be an open set. Then

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

Now, we know that $g^{-1}(U)$ is open, and thus $f^{-1}(g^{-1}(U))$ is also open which completes the proof. □

Definition 26. *Let X, Y be metric spaces. The map $f : X \to Y$ is called an **Isometry** if for all $x_1, x_2 \in X$, we have*

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

Example 11. *Parseval's Identity*
If we take $f \in C([0, 2\pi])$ and we define

$$a_0 = 0, \qquad a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin(nx)dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(nx)dx$$

then

$$\int_0^{2\pi} |f(x)|^2\, dx = ||f||_2^2 = C \cdot \left(\sum_{n=0}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$$

$$f \to \{\underline{a} = (a_0, \ldots, a_n, \ldots), \underline{b} = (b_1 \ldots, b_n, \ldots)\}$$

or we could write

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)[\cos(nx) + i\sin(nx)]dx$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}\, dx$$

for

$$f \to \{(a_0, a_1 + ib_1, \ldots, a_n + ib_n, \ldots\}$$

and so

$$||f||_2^2 = C \cdot (||\underline{a}||_{l^2}^2 + ||\underline{b}||_{l^2}^2)$$

Thus, we have

$$||(\underline{a}, \underline{b}|| = ||f||_{L^2} = \frac{\sqrt{||\underline{a}||_2^2 + ||\underline{b}||_2^2}}{\sqrt{C}}$$

DEFINITION 27. *Let X, Y be topological spaces. The map $f : X \to Y$ is a **Homeomorphism** if*

(i) f is bijective.

(ii) Both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous funcitons.

*We say that X, Y are **Homeomorphic** if and only if there exists a homeomorphism $f : X \to Y$. Then open/closed sets, closure, limit points and boundary are all the same for X and Y. Also, continuous functions on X, Y are the same.*

Chapter 3

Completeness

Section 3.1

Basics & Definitions

Definition 28. *A sequence (x_n) in a metric space X is a **Cauchy Sequence** if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n > N$, then*

$$d(x_n, x_m) < \epsilon$$

Definition 29. *We say that a metric space X is **Complete** if and only if every Cauchy sequence is convergent. That is, if (x_n) is Cauchy, then there exists $z \in X$ such that*

$$d(z, x_n) \to 0$$

as $n \to \infty$.

Example 12. *Examples of Complete Metric Spaces*

- *The space l^2. Now, l^2 is the space of sequences. Let $\underline{x}^1, \dots, \underline{x}^k, \dots \in l^2$ where*

$$\underline{x}^k = (x_1^k, \dots, x_n^k, \dots) \in l^2$$

and such that

$$\sum (x_j^k)^2 < \infty$$

Now, suppose that the sequence $(\underline{x}^k)$ is Cauchy in l^2 which is equivalent to

$$||\underline{x}^n - \underline{x}^m||_2 \to 0$$

as $m, n \to \infty$. We want to find

$$\underline{y} = (y_1, \dots, y_n, \dots) \in l^2$$

such that $||\underline{x}^k - \underline{y}||_2^2 \to 0$ as $k \to \infty$.
We know that

$$||\underline{x}^k - \underline{x}^l||_2^2 = \sum_{j=1}^{\infty} (x_j^k - x_j^l)^2 \geq (x_m^k - x_m^l)^2$$

For each m, (x_m^k) is cauchy in $\mathbb{R}$. Now there exists $y_m \in \mathbb{R}$ such that $x_m^k \to y_m$ as $k \to \infty$. The sequence y_j is forced upon us. So, let

$$\underline{y} = (y_1, \dots, y_n, \dots)$$

and we claim now that $\underline{y} \in l^2$. We let $\epsilon > 0$ and let N_1 be such that $||\underline{x}^n - \underline{x}^m||_2^2 < \epsilon$ if $m, n > N_1$, and we get

$$\sum_{j=1}^{\infty} (x_j^m - x_j^n)^2 = \sum_{j=1}^{N_2} (x_j^m - x_j^n)^2 + \sum_{j=N_2+1}^{\infty} (x_j^m - x_j^n)^2$$

and the first sum is $\leq \epsilon$ and the second sum is $\leq \epsilon$. Fix n, and let $m \to \infty$. As $m \to \infty$, $x_j^m \to y_j$. This implies that

$$\sum_{j=1}^{N_2} (y_j - x_j^n)^2 \leq \epsilon$$

$M - 2$ is any natural number. let $M_2 \to \infty$. This gives

$$\sum_{j=1}^{\infty} (y_j - x_j^n) < \infty$$

Now, if $(\underline{x}^n) \in l^2$, we have that $d^2(\underline{y}, \underline{x}^n) \leq \epsilon$ and thus $\underline{y} \in l^2$. Now

$$\sqrt{\sum y_j^2} \leq \sqrt{\sum x_j^2} + \sqrt{\sum (x_j^n - b_j)^2} \leq \sqrt{\epsilon}$$

and so

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} (x_j^n - b_j)^2 \to 0$$

which follows from the inequality

$$\sum_{j=1}^{\infty} (x_j^n - y_j)^2 \leq \epsilon$$

provided that $n > M_1$ and the fact that ϵ is arbitrary yields our claim. □

Excercise. Show that l^p is complete.
Let's examine $C([a,b])$ with d_∞ distance

$$d_\infty(f, g) = ||f - g||_\infty = \max_{x \in [a,b]} |f(x) - g(x)|$$

Theorem 30. *The space $(C([a,b]), d_\infty)$ is complete.*

Proof.
Let $f_1(x), \ldots, f_n(x), \ldots$ be a Cauchy sequence in $C([a,b])$. Fix $x_0 \in [a,b]$. Then $(f_n(x_0))$ is a Cauchy sequence.

$$|f_i(x_0) - f_j(x_0)| \leq \max_{x \in [a,b]} |f_i(x) - f_j(x)| \to 0$$

as i, j simultaneously approach ∞. Therefore, $(f_n(x_0))$ converges to a limit that we call $g(x_0)$. It is clear that

$$d_\infty(f_i(x), g(x)) \to 0$$

as $i \to \infty$.
Let $\epsilon > 0$, let $N \in \mathbb{N}$ be such that

$$d_\infty(f_n, f_m) < \epsilon$$

for $n, m > N$.

$$\Longrightarrow \forall\, x \in [a,b], \quad d(f_n(x), f_m(x)) < \epsilon$$

Let $m \to \infty$, then $f_m(x) \to g(x)$. Now, passing to the limit, we get that

$$\Longrightarrow |f_n(x) - g(x)| \leq \epsilon$$

if $n > N$, then

$$d_\infty(f_n, g) \leq \epsilon$$

It now remains to show that $g(x)$ is continuous. Fix $\epsilon > 0$ and choose $N > 0$ such that

$$d_\infty(f_n, f_m) < \frac{\epsilon}{2}$$

for each $m \geq N$. Now, let $f_N(x)$ be uniformly continuous on $[a, b]$ (we can do this since $[a, b]$ is a compact subset of $\mathbb{R}$). Let $\delta > 0$ be such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

Let $m > N$. Suppose that $f_m(x) \to g(x)$. As before, let $|x - y| < \delta$, then

$$|g(y) - g(x)| \leq |g(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - g(x)| < \frac{3\epsilon}{2} \qquad \square$$

Theorem 31. *Let $1 \leq p < \infty$. Then $(C([a, b]), d_p)$ is not complete.*
Proof.
We shall define a Cauchy sequence

$$f_n(x) \to g(x) = \begin{cases} 0, & x \in [-1, 0] \\ 1, & x \in (0, 1] \end{cases}$$

Let

$$f_n(t) = \begin{cases} nt, & t \in \left[0, \frac{1}{n}\right] \\ 1, & t > 1\frac{1}{n} \end{cases}$$

Now,

$$\int_0^{1/n} f_n(t)dt = \int_0^{1/n} nt\ dt = \left[\frac{nt^2}{2}\right]_0^{1/n} = \frac{1}{2n} \to 0$$

as $n \to \infty$. We prove now for $p = 1$. Let

$$f_n(x) = \begin{cases} 0, & x \in \left[-1, \frac{1}{2n}\right] \\ 2^n \left(x - \frac{1}{2n}\right), & x \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \\ 1, & x \in \left[\frac{1}{2^n}, 1\right] \end{cases}$$

so then, for $m > n$

$$f_m(x) - f_n(x) = \begin{cases} 0, & x \leq \frac{1}{2^{m+1}} \\ 2^m \left(x - \frac{1}{2^{m+1}}\right), & x \in \left[\frac{1}{2^{m+1}}, \frac{1}{2^m}\right] \\ 1, & x \in \left[\frac{1}{2^m}, \frac{1}{2^{n+1}}\right] \\ 1 - 2^n \left(x - \frac{1}{2^{n+1}}\right), & x \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \end{cases}$$

Now,

$$d_1(f_m, f_n) = \int_{1/2^{m+1}}^{1/2^n} |f_m(x) - f_n(x)|dx \leq \left(\frac{1}{2^n} - \frac{1}{2^{m+1}}\right) \cdot 1 \to 0$$

as $m, n \to \infty$ $\qquad \square$.

Exercise. Let

$$h_{n,p}(x) = \begin{cases} 0, & x \in [-1, 0] \\ (nx)^{1/p}, & x \in \left[0, \frac{1}{n}\right] \\ 1, & x \in \left[\frac{1}{n}, 1\right] \end{cases}$$

and

$$\left[\int_0^{1/n} |h_{n,p}(x)|^p dx\right]^{1/p} = \left[\int_0^{1/n} (|nx|^{1/p})^p dx\right]^{1/p} = \left[\int_0^{1/n} (nx) dx\right]^{1/p} = \left(\frac{1}{2n}\right)^{1/p} \to 0$$

as $n \to \infty$. Use $h_{n,p}(x)$ to modify the construction for $p = 1$.

Theorem 32. *The space X is complete if and only if for every sequence of nested closed balls*

$$\cdots \subset B(x_n, r_n) \subset \cdots \subset B(x_1, r_1)$$

such that $r_n \to 0$ as $n \to \infty$. We then have a nonempty intersection.

Proof.

($\Longrightarrow$) *Let (x_n) be a sequence of centres of the balls. Then, (x_n) is a Cauchy sequence. Indeed,*

$$d(x_n, x_m) \leq \max(r_n, r_m) \to 0$$

as $n, m \to \infty$. X is complete, so $x_n \to y \in X$ as $n \to \infty$. We want to show that

$$y \in \bigcap_{i=n}^{1} B(x_i, r_i)$$

We know that

$$y = \lim_{n\to\infty} x_n$$

for $x_m \in B(x_n, r_n)$, $m > n$, we get

$$\lim_{n\to\infty} d(x_m, x_n) \leq r_n \Longrightarrow d(y, x_n) \leq r_n$$

($\Longleftarrow$) *We must prove now that if X is not complete, then $\exists$ a sequence of balls*

$$\cdots \subset B(x_n, r_n) \subset \cdots \subset B(x_1, r_1)$$

such that $r_n \downarrow 0$ and

$$\bigcap_{n=1}^{\infty} B(x_n, r_n) = \emptyset$$

Suppose now that X is not complete. Then there exists a Cauchy sequence (x_n) such that x_n doesn't have a limit in X. Also, there exists n_1 such that

$$d(x_m, x_{n_1}) < \frac{1}{2}$$

for $m \geq n_1$. Now, let $B_1 = B(x_{n_1}, 1)$, then there exists $n_2 > n_1$ such that

$$d(x_m, x_{n_2}) < \frac{1}{2^2} = \frac{1}{4}$$

for each $m \geq n_2$. Now let $B_2 = B(x_{n_2}, 1/2)$. We claim that $B_2 \subset B_1$.
The induction step is that there exists $n_k > n_{k-1}$ such that

$$d(x_m, x_{n_k}) < \frac{1}{2^k}$$

for all $m \geq n_k$. Now, let

$$B_k = B\left(x_{n_k}, \frac{1}{2^k}\right)$$

Now, we claim that $B_k \subset B_{k-1}$.
We know that

$$\cdots \subset B_2 \subset B_1$$

where

$$r_k = \frac{1}{2^k} \to 0$$

as $k \to \infty$. Any point lying in

$$\bigcap_{k=1}^{\infty} B_k$$

must be a limit of (x_k). We assumed that (x_k) doesn't converge. Thus

$$\bigcap_{k=1}^{\infty} B_k = \emptyset$$

and this contradiction completes the $(\Longleftarrow)$ direction of the proof. $\square$

In general, there exist complete metric spaces, and there exists r_n that doesn't converge to zero such that

$$\cdots \subset B(x_2, r_2) \subset B(x_1, r_1)$$

but that

$$\bigcap_{k=1}^{\infty} B(x_k, r_k) = \emptyset$$

Hint (Problem 5(ii), Assignment 1): Prove that

$$\{\text{All 3-adic rationals in } [0,2]\} \subset K + K$$

and then $a/3^n$ is dense in $[0,2]$ where $0 \leq a \leq 2 \cdot 3^n$.

Section 3.2

Completion & Density Revisited

Definition 30. *The **Completion** of an incomplete metric space X is a metric space Y such that*

$$f : X \to Y$$

is a function that satisfies

- *f is 1-to-1, with*

  $$d(f(x_1), f(x_2)) = d(x_1, x_2)$$

  [f is an isometry from X to $f(X)$]
- *$f(X)$ as dense in Y*
- *Y is complete*

Proposition 33. *Every incomplete metric space has a completion that is unique up to isometry.*
Proof (Idea). *Let Z be the set of all Cauchy Sequences $\underline{x} = (x_1, \ldots, x_n, \ldots)$ in X.*

Definition 31. *We say that two Cauchy sequences* (x_n) *and* (y_n) *are* ***Equivalent*** *if and only if*

$$d(x_n, y_n) \to 0$$

as $n \to \infty$.

Proposition 34. *Equivalence is an equivalence relation.*

Proof.

The only non-trivial part of this part is transitivity. Thus, if $(x_n) \sim (y_n)$ *and* $(y_n) \sim (z_n)$*, then we know that for* $n > n_1$,

$$d(x_n, y_n) < \frac{\epsilon}{2}$$

and for $n > n_2$,

$$d(y_n, z_n) < \frac{\epsilon}{2}$$

and thus

$$d(x_n, z_n) < \epsilon$$

for $n > N = \max\{n_1, n_2\}$. $\square$

Definition 32. *If we let* Y *denote the set of all equivalence classes in* Z *with respect to* $\sim$*, then for* $(x_n), (y_n)$ *Cauchy sequences in* X*, we define*

$$d\left(\overline{(x_n)}, \overline{(y_n)}\right) = \lim_{n\to\infty} d_X(x_n, y_n)$$

Proposition 35.

$$d = 0 \Longleftrightarrow (x_n) \sim (y_n)$$

Proof.

So, if $(x_n) \sim (x'_n)$*, then*

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n)$$

We have

$$f : X \to Y$$

such that $f(x) = \{x, x, x, \ldots\}$*. It remains to show that*

- $f(X)$ *is dense in* Y
- Y *is complete*

For the first part, let $(x_n^1), (x_n^2), \ldots, (x_n^k)$ *be Caucy sequences. We know that*

$$d_Y((x_n^{k_1}), (x_n^{k_2})) \to 0$$

as $k_1, k_2 \to infty$*. Now, let* $y \in Y$*.* y *is an equivalence class of the sequence* (x_n)*. Let* $\epsilon > 0$*, and let* N *be such that*

$$d(x_n, x_m) < \epsilon$$

for all $n, m > N$*. Choose,* $n > N$*, and we get*

$$d_Y(y, f(x_n)) = d_Y(y, (x_n, x_n, \ldots, x_n, \ldots)) \leq \epsilon$$

where $f(x_n) \in f(X)$*. This implies that* $f(X)$ *is dense in* Y*.*
For the second part, let $y_1, \ldots, y_n$ *be Cauchy in* Y*. Choose*

$$\left\{x_n : d_X(f(x_n), y_n) < \frac{1}{n}\right\}$$

We claim that (x_n) *is a Cauchy sequence in* X*. If* $z = \overline{(x_n)}$*, then* $y_n \to z$ *in* Y*.*

We now ask the question: Why are complete metric spaces cool?
A possible answer would involve the Contraction Mapping Principle.

Section 3.3

O.D.E. & Contraction

Definition 33. *If X is a metric space and $A : X \to X$, then we say that A is a* ***Contraction Mapping*** *if there exists $0 < \alpha < 1$ such that*

$$d(A(x), A(y)) < \alpha \cdot d(x, y)$$

for any $x, y \in X$. It is a fact that if A is a contraction mapping, then A is continuous.

Theorem 36 (Contraction Mapping Principle). *If X is complete, and A is a contraction mapping, then there exists a unique point $x_0 \in X$ such that $A(x_0) = x_0$ is a fixed point of A. Moreover, for any $x \in X$, $A^n(x) \to x_0$ as $n \to \infty$.*

Proof.

Let $x \in X$ and consider

$$\{x, A(x), \ldots, A^n(x), \ldots\} = \{x_0, \ldots, x_n, \ldots\}$$

We claim that (x_n) is a Cauchy sequence. To this end, we begin with

$$d(A^m(x), A^{m+k}(x)) = d(A^m(x), A^m(A^k(x))) \leq \alpha^m d(x, A^k(x))$$

By induction on m, we see

$$d(A^2(x), A^2(y)) \leq \alpha d(A(x), A(y)) \leq \alpha^2 d(x, y) \leq \cdots$$

and so if $m \to \infty$, then $\alpha^m \to 0$.
Now, we know that $A^n(x) \to x_0$ as $n \to \infty$, since X is complete. We now want an a priori bound on $d(x, A^k(x))$ for each k.

$$d(x, A(x)) = b$$

$$d(x, A^k(x)) \leq d(x, A(x)) + d(A(x), A^2(x)) + \cdots + d(A^{k-1}(x), A^k(x))$$

Where

$$d(A^i(x), A^{i-1}(x)) \leq \alpha^i b$$

and so

$$d(x, A^k(x)) \leq b(1 + \alpha + \cdots + \alpha^{k-1}) \leq \frac{b}{1-\alpha}$$

which is the bound we need. We also know that

$$A^{n+1}(x) \to A(A^n(x)) \to A(x_0)$$

as $n \to \infty$. And thus $A(x_0) = x_0$ which means that x_0 is a fixed point of A. A cannot have two fixed points since if x_0, y_0 are fixd, then

$$Ax_0 = x_0 \qquad A(y_0) = y_0$$

then

$$d(A(x_0), A(y_0)) = d(x_0, y_0)$$

which is a contradiction that completes the proof. $\square$

Remark. Alternatively to our original proof, we let our sequence be $x_n = Ax_{n-1}$ so that $x_n = A^n x_0$ and since A is continuous, and since $x_n \to x^*$, we get that

$$Ax^* = A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n = x^*$$

which shows existence of a fixed point.

Definition 34. *We say that a map f is **Lipschitz**, denoted $f \in Lip_K$ if*

$$|f(x) - f(y)| < K \cdot |x - y|$$

and if $K < 1$ then f is a contraction mapping.

Example 13.

1. *Suppose that $f : [a, b] \to \mathbb{R}$ and that f is Lipschitz with $K > 0$. If $K < 1$ then we know that f is a contraction mapping from $[a, b] \to [a, b]$. Hence if*

   $$x_0, x_1 = f(x_1), x_2 = f(f(x_0)), \ldots$$

   then $f^n(x) \to y$ such that $f(y) = y$. Sufficient condition is that if $f \in C^1([a, b])$,

   $$|f'(t)| \leq K < 1$$

   for all $t \in [a, b]$.

   $$f(y) - f(x) = (y - x) \cdot f'(t)$$

   for some $t \in [x, y] \subset [a, b]$. So

   $$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(t)| \leq K < 1$$

2. *Solving ODE's.*

   $$\frac{d}{dx} y = f(x, y) \qquad y(x_0) = y_0$$

   Now, suppose that

   $$|f(x, y_1) - f(x, y_2)| \leq M \cdot |y_1 - y_2|$$

   for M fixed, then $f \in Lip_M$ in y. We assume that f is continuous on some rectangle $R \subset \mathbb{R}^2$ such that $(x_0, y_0) \in R$. Then, on some small interval

   $$|x - x_0| \leq d$$

   there exists a unique solution to the ODE satisfying the initial condition $y(x_0) = y_0$.
   Proof.
   Solving the ODE above is equivalent to solving an integral equation

   $$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt \qquad \varphi(x_0) = y_0$$

   and we note that

   $$\varphi'(x) = f(x, \varphi(x))$$

   f is continuous on R and so

   $$|f(x, y)| \leq K$$

   for all points $(x, y) \in R_1$ where R_1 contains (x_0, y_0). Now we choose $d > 0$ so that if

   - *$|x - x_0| \leq d$ and $|y - y_0| \leq Kd$, and*

- $Md < 1$

then $(x, y) \in R_1$.
Let C *be the space of continuous functions* φ *on* $[x_0 - d, x_0 + d]$ *such that*

$$|x - y_0| \leq K \cdot d$$

Let d_∞ *be the* sup *distance*

$$d(\varphi_1, \varphi_2) = \sup_{x \in [x_0 - d, x_0 + d]} |\varphi_1(x) - \varphi_2(x)|$$

then C *is complete. Now we need to define a contraction mapping and so let*

$$A\varphi = \psi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt$$

So we claim that A *is a contraction map from* $C \to C$. *If this claim is true, then there exists a unique fixed point* φ *of* A *such that* $A\varphi = \varphi$ *which means that this unique* φ *solves the integral equation uniquely and thus solves the ODE uniquely.*
Now, to prove the claim, we define a starter function $\varphi \in C$ *such that*

$$|x - x_0| \leq d$$

and we have that

$$\begin{aligned}
|\psi(x) - y_0| = |A(\varphi(x)) - y_0| &= \left| \int_{x_0}^{x} f(\varphi(t), t)\, dt \right| \\
&\leq \max_{t \in [x_0, x]} |f(\varphi(t), t)| \cdot |x - x_0| \\
&\leq Kd
\end{aligned}$$

Now, for the contraction (with respect to the sup *norm)*

$$\begin{aligned}
|A(\varphi_1(x)) - A(\varphi_2(x))| &= \left| y_0 + \int_{x_0}^{x} f(t, \varphi_1(t))\, dt - y_0 - \int_{x_0}^{x} f(t, \varphi_2(t))\, dt \right| \\
&= \left| \int_{x_0}^{x} [f(t, \varphi_1(t)) - f(t, \varphi_2(t))]\, dt \right| \\
&\leq \int_{x_0}^{x} |f(t, \varphi_1(t)) - f(t, \varphi_2(t))|\, dt \\
&\leq \int_{x_0}^{x} M \cdot |\varphi_1(t) - \varphi_2(t)|\, dt \\
&\leq M \cdot |x - x_0| \cdot \max_{t \in [x_0, x]} |\varphi_1(t) - \varphi_2(t)| \\
&\leq M \cdot d \cdot d_\infty(\varphi_1, \varphi_2)
\end{aligned}$$

Thus,

$$d_\infty(A(\varphi_1), A(\varphi_2)) \leq M \cdot d \cdot d_\infty(\varphi_1, \varphi_2)$$

remember that $M \cdot d < 1$ *and* $M = \alpha$.

Fact. Let X be a complete metric space. If A^n is a contraction map from X. Then $A(x) = x$ also has a unique solution.
We will leave the proof of this until when we need to use it.

Chapter 4

Compactness

Section 4.1

Basics & Definitions

Definition 35. *Let $A \subset X$ where X is a metric space. We say that A is* ***Sequentially Compact*** *if every sequence in A has a subsequence which converges to some $x \in X$ ($x \notin A$ is possible).*

Definition 36. *$A \subset X$ is called an* ***ϵ-net*** *if for each $x \in X$, there exists $y \in A$ such that*

$$d(x, y) \leq \epsilon$$

Definition 37. *X is* ***Totally Bounded*** *if for each $\epsilon > 0$, there exists a finite ϵ-net in X.*
OR
X is ***Totally Bounded*** *if for any $\epsilon > 0$, there exists a finite set $x_1, \ldots, x_n$ such that*

$$X \subset B(x_1, \epsilon) \bigcup B(x_2, \epsilon) \bigcup \cdots \bigcup B(x_n, \epsilon)$$

where $n = n(\epsilon)$.

Other problems where knowing what $n(\epsilon)$ is useful include

- Coding
- Complexity

We note first that a totally bounded space is bounded. That is,

$$d(x_1, x_2) \leq 2\epsilon + \max_{y_1, y_2 \in \epsilon - net\ A} d(y_1, y_2) = 2\epsilon + diam(\epsilon - \text{net } A)$$

Fact. If $B \subset X$ is totally bounded, then $\overline{B} \subset X$ is also totally bounded.

We now ask the question: What is the minimal number of points in an ϵ-net? $[0,1]^n$,

$$\text{least number of points} \approx \left(\frac{1}{\epsilon} + 1\right)^n$$

Hint: We can use without proof the fact that all norms in $\mathbb{R}^2$ are equivalent. This implies that we can use our favourite norm in $\mathbb{R}^n$ to define the operator norm, namely

$$\frac{||Ax||_p}{||x||_p} \quad \text{or} \quad \frac{||Ax||_\infty}{||x||_\infty}$$

Now, suppose that $C_1 < C_2$, so if

$$\frac{1}{C_1} \leq \frac{d_1(x,y)}{d_2(x,y)} < C_1$$

then

$$\frac{1}{C_2} \leq \frac{d_1(x,y)}{d_2(x,y)} < C_2$$

Example 14. *Examples of totally bounded sets.*

- *In $\mathbb{R}^n$, a set is totally bounded iff the set is bounded (n is fixed).*
- *In l^2, take the set*

$$A = \left\{ x = (x_1, \ldots, x_n, \ldots) : |x_1| \leq 1, |x_2| \leq \frac{1}{2}, \cdots |x_n| \leq \frac{1}{2^n} \right\}$$

We claim that A is totally bounded.
Proof.
Let $\epsilon > 0$ and choose m such that

$$\frac{1}{2^m} < \frac{\epsilon}{2}$$

Now, let

$$C_1 = \{\underline{x} = (x_1, \ldots, x_m, 0, 0, \ldots, 0, \ldots) : \underline{x} \in A\}$$

First, we claim that C_1 is totally bounded. C_1 can be covered by $N(\epsilon) < \epsilon^{-m}$ balls of radius ϵ.
Now we claim that the whole set C lies in an ϵ-neighbourhood of C_1. It suffices to show that for any $x \in C$, there exists $y \in C_1$ such that

$$d(x,y) \leq \epsilon$$

To this end, take $x = (x_1, \ldots, x_m, x_{m+1}, \ldots) \in C$ and $y = (x_1, \ldots, x_m, 0, \ldots, 0, \ldots) \in C_1$. Now,

$$\begin{aligned} d^2(x,y) = \sum_{k=m+1}^{\infty} |x_k^2 &\leq \frac{1}{(2^{m+1})^2} + \frac{1}{(2^{m+2})^2} + \cdots \\ &= \frac{1}{4^m}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= \frac{1}{4^m}\frac{1}{1 - 1/4} = \frac{1}{3 \cdot 4^{m-1}} < \epsilon^2 \end{aligned}$$

Now, the ϵ-net in C_1 is also a finite (2ϵ)-net in C. So since, ϵ is arbitrary, we conclude that C is totally bounded.
We remark that the same construction would work if

$$|x_k| \leq a_k \ \ s.t. \ \sum_{k=1}^{\infty} a_k^2 < \infty$$

- *We claim that the whole of l^2 is not totally bounded. To this end, define*

$$x_1 = (1, 0, 0, 0, \ldots)$$

$$x_2 = (0, 1, 0, 0, \ldots)$$

$$x_3 = (0, 0, 1, 0, \ldots)$$

$$A = \{x_1, x_2, \ldots\}$$

We know that

$$d^2(x_m, x_n) = 1^2 + 1^2 = 2$$

for any m, n. *So, for any* $\epsilon < \sqrt{2}/2$, A *cannot be covered by a finite* ϵ*-net.*

THEOREM 37. *Let* X *be complete, and suppose that* $A \subset X$. A *is sequentially compact if and only if* A *is totally bounded.*

Proof.

$(\Longrightarrow)$ *Suppose that* A *is not totally bounded. There exists an* $\epsilon > 0$ *such that* A *doesn't have a finite* ϵ*-net. Choose* $x_1 \in A$, *then there exists* $x_2 \in A$ *such that*

$$d(x_1, x_2) \geq \epsilon$$

and there exists x_3 *such that*

$$d(x_3, x_1) \geq \epsilon \quad d(x_3, x_2) \geq \epsilon$$

and continuing this way, we see that there exists x_k *such that for any* $j < k$,

$$d(x_k, x_j) \geq \epsilon$$

Now, we claim that x_j *cannot have a convergent subsequence. To yield this claim, we simply say that any subsequence is not Cauchy.*

$(\Longleftarrow)$ *Suppose that* X *is complete and* A *is totally bounded. Let* (x_a) *be a sequence of points in* A. *Let*

$$\epsilon_1, \epsilon_2 = \frac{1}{2}, \ldots, \epsilon_k = \frac{1}{k}$$

For any k, *there exists a finite set* $a_1^k, a_2^k, \ldots, a_{n_k}^k$ *such that*

$$\bigcup_{i=1}^{n} B(a_i^k, \epsilon_k) = A$$

One of $B(a_j^1, 1)$ *has infinitely many* x_k*'s. In a ball* B_i *of radius 1, there is an infinite subsequence of* x_k*'s*

$$x_1^{(1)}, \ldots, x_n^{(1)}, \ldots$$

Also, one of the balls

$$B\left(a_j^2, \frac{1}{2}\right)$$

has infinitely many $x_j^{(1)}$. *Call it* B_2. *Call the corresponding subsequence*

$$x_1^{(2)}, \ldots, x_n^{(2)}, \ldots$$

One of

$$B\left(a_j^k, \frac{1}{k}\right)$$

has infinitely many points of

$$x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \ldots$$

Call it B_k. *Elements lying in* B_k *for a subsequence*

$$x_1^{(k)}, \ldots, x_n^{(k)}, \ldots$$

Now, let $y_k = x_k^{(k)}$. y_k *is a subsequence of* x_k*'s.*

y_k *is Cauchy. For* $m \geq 1$,

$$d(y_k, y_{k+m}) \leq diam\left(B\left(a_j^k, \frac{1}{k}\right)\right) \leq \frac{2}{k}$$

X *is complete.* $y_k \to z \in X$. $\square$

ARZELA & COMPACTA

PROPOSITION 38. *X is complete, and $A \subseteq X$ is compact if and only if for all $\epsilon > 0$, there exists in X a sequentially compact ϵ-net that covers A.*

DEFINITION 38. *Let $\mathcal{F} = \{\varphi(x)\}$ be a family (collection) of functions on $[a, b]$. We say that $\mathcal{F}$ is* ***Uniformly Bounded*** *if there exists $M > 0$ such that*

$$|\varphi(x)| \leq M, \ \ \forall\, x \in [a, b], \ \ \forall\, \varphi \in \mathcal{F}$$

DEFINITION 39. *We say that the family $\mathcal{F}$ is* ***Equicontinuous*** *if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\forall\, x_1, x_2 \in [a, b]$, with $|x_1 - x_2| < \delta$, and $\forall\, \varphi \in \mathcal{F}$, we get*

$$|\varphi(x_1) - \varphi(x_2)| < \epsilon$$

Remark. Both of the above definitions work for any metric space X.

THEOREM 39 (Arzela). *Let $\mathcal{F}$ be a family of continuous functions on $[a, b]$. Then $\mathcal{F}$ is sequentially compact in $(C([a, b]), d_\infty)$ (which is complete by Theorem 30) if and only if $\mathcal{F}$ is uniformly bounded and equicontinuous.*
Proof.
($\Longrightarrow$) It is true that $\mathcal{F}$ is sequentially compact in $(C([a, b]), d_\infty)$ if and only if $\mathcal{F}$ is totally bounded in $C([a, b])$ by Theorem 36.
Now, let $\epsilon > 0$, then $\mathcal{F}$ can be covered by a finite $(\epsilon/3)$-net. There exists $\varphi_1, \ldots, \varphi_k \in C([a, b])$ such that for any $\varphi \in C([a, b])$, we have

$$d_\infty(\varphi, \varphi_j) < \frac{\epsilon}{3}$$

for some $1 \leq j \leq k$. Now, $|\varphi_j(x)| \leq M_j$ for all $x \in [a, b]$. Take

$$M = \max\{M_j\} + \frac{\epsilon}{3}$$

and for any $x \in [a, b]$ and for any $\varphi \in \mathcal{F}$, there exists $1 \leq j \leq k$ such that

$$|\varphi(x) - \varphi_j(x)| \leq \frac{\epsilon}{3}$$

Now, for all x,

$$|\varphi(x)| \leq |\varphi_j(x)| + \frac{\epsilon}{3} \leq M_j + \frac{\epsilon}{3} \leq M$$

and since

$$d_\infty(\varphi, \varphi_j) \leq \frac{\epsilon}{3}$$

which means that $\mathcal{F}$ is uniformly bounded.
Each $\varphi(x) \in C([a, b])$ is uniformly continuous, so there exists δ_j such that

$$|x_1 - x_2| < \delta_j \Longrightarrow |\varphi_j(x_1) - \varphi_j(x_2)| < \frac{\epsilon}{3}$$

and let $\delta = \min\{\delta_j\}$. Let $x_1, x_2 \in [a, b]$ such that $|x_1 - x_2| < \delta$. Let $\varphi \in \mathcal{F}$ and choose $1 \leq j \leq k$ so that

$$d_\infty(\varphi, \varphi_j) < \frac{\epsilon}{3}$$

and now

$$\begin{aligned}|\varphi(x_1) - \varphi(x_2)| &\leq |\varphi(x_2) - \varphi_j(x_1)| + |\varphi_j(x_1) - \varphi_j(x_2)| + |\varphi_j(x_2) - \varphi(x_2)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

($\Longleftarrow$) *We know that $(C([a,b]), d_\infty)$ is complete and hence it is sequentially compact if and only if it is totally bounded. We claim that $\mathcal{F}$ is totally bounded. Let $\epsilon > 0$. By uniform boundedness, we get that*

$$|\varphi(x)| \leq M$$

Now, let $\delta > 0$, then for any $x_1, x_2 \in [a,b]$, we have that

$$|x_1 - x_2| < \delta \Longrightarrow |\varphi(x_1) - \varphi(x_2)| < \frac{\epsilon}{5}$$

for any $\varphi \in \mathcal{F}$. Now, let

$$\frac{b-a}{N} < \delta$$

and let

$$x_0 = a \quad x_1 = a + \frac{b-a}{N} \quad x_2 = a + 2\frac{b-a}{N} \quad \cdots \quad x_{N-1} = a + (N-1)\frac{b-a}{N} \quad x_N = b$$

Now we let $m > 0$ and

$$\frac{2M}{m} < \frac{\epsilon}{5}$$

and $y_0 = -M$ and $y_m = M$ so that the points y_k subdivide the interval $[-M, M]$ into m equal parts. Now, we take any continuous function $\varphi \in \mathcal{F}$. We look at values $\{\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_N)\}$ and for each j, we choose y_j such that

$$|\varphi(x_j) - y_j| \leq \frac{\epsilon}{5}$$

Now let ψ be a piecewise-linear function such that $\psi(x_j) = y_j$. The set of all possible ψ (call it A) is finite and

$$|A| \leq (M+1)^{n+1}$$

Furthermore, since we know that the slope is between -3 and 3, it follows that the number of functions

$$|A| \leq (M+1) \cdot 7^n$$

Now we claim that the set A of possible ψ form an ϵ-net in $\mathcal{F}$. To this end, we first note that for any $0 \leq k \leq n$,

$$|\varphi(x_k) - \psi(x_k)| < \frac{\epsilon}{5}$$

and

$$|\varphi(x_{k+1}) - \psi(x_{k+1})| < \frac{\epsilon}{5}$$

by our choice of ψ. Also, by uniform continuity, we get

$$|\varphi(x_k) - \varphi(x_{k+1})| < \frac{\epsilon}{5}$$

$$\Longrightarrow |\psi(x_k) - \psi(x_{k+1})| < \frac{3\epsilon}{5}$$

Now, for any $x \in [x_k, x_{k+1}]$, we have by linearity that

$$|\psi(x) - \varphi(x_k)| \leq \frac{3\epsilon}{5}$$

Then

$$|\varphi(x) - \psi(x)| \leq |\varphi(x) - \varphi(x_k)| + |\varphi(x_k) - \psi(x_k)| + |\psi(x_k) - \psi(x)| = \alpha + \beta + \gamma$$

We know that $\alpha < \epsilon/5$ by uniform continuity, and $\beta < \epsilon/5$ by construction of ψ and $\gamma < 3\epsilon/5$. Thus

$$|\varphi(x) - \psi(x)| < \epsilon$$

for any $x \in [a, b]$ which completes the proof. $\square$.

Recall that $A \subseteq X$ is sequentially compact if and only if any sequence $(x_n) \in A$ has a subsequence that converges in X.

DEFINITION 40. *A subset $A \subseteq X$ is **Sequentially Compact In Itself** if and only if any sequence $(x_n) \in A$ has a convergent subsequence that converges in A.*

EXAMPLE 15. *If $B = \mathbb{Q} \cap [0, 1]$ then B is sequentially compact but not in itself.*

EXAMPLE 16. *If $A = X$ then sequential compactness is equivalent to sequential compactness in itself.*

DEFINITION 41. *A sequentially compact metric space is called **Compactum**.*

PROPOSITION 40. *Let $A \subseteq X$. If A is sequentially compact, then A is sequentially compact in itself if and only if A is as a closed subset of X.*

COROLLARY 41. *Any closed bounded subset of $\mathbb{R}^n$ is sequentially compact in itself.*

PROPOSITION 42. *Let X be a metric space. X is a compactum if and only if X is complete and totally bounded.*
Proof.
EXCERCISE

PROPOSITION 43. *Every compactum has a countable dense subset.*

THEOREM 44. *The following are equivalent:*

(i) X is a compactum.

(ii) An arbitrary open cover $\{U_\alpha\}_{\alpha \in I}$ of X has a finite subcover. That is there exists $\alpha_1, \ldots, \alpha_n \in I$ such that

$$X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

(iii) (Finite Intersection Property)
A family $\{F\}_{\alpha \in I}$ of closed subsets of X such that every finite collection of F_n's has a nonempty intersection has

$$\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$$

Proof.
($(i) \iff (ii)$) Suppose that X is sequentially compact in itself. Let $\epsilon_n = 1/n$ and take a finite ϵ_n-net; with

centres $a_k^{(n)}$ *for each* n *so that*

$$X \subseteq \bigcup B\left(a_k^{(n)}, \frac{1}{n}\right)$$

We proceed by contradiction. Assume that there exists an open cover $\{U_\alpha\}$ *without a finite subcover and choose one of*

$$B\left(a_k^{(n)}, \frac{1}{n}\right)$$

that cannot be covered by a finitely many U_α*'s. Now, say*

$$B\left(a_{k_0(n)}^{(n)}, \frac{1}{n}\right)$$

Let $x_n = \{x_{k_0(n)}^{(n)}\}$*. Now,* X *is a compactum, so a subsequence* $x_{n_j} \to y \in X$*.* $y \in U_\beta$*, for some* $\beta \in I$*.* U_β *is open and so*

$$B(y, \epsilon) \subseteq O_\beta$$

for some $\epsilon > 0$*. Now choose* n *so that*

$$\frac{1}{n} < \frac{\epsilon}{2}$$

and then

$$d(y, a_{k_0(n)}^{(n)}) < \frac{\epsilon}{2}$$

Now, we claim that

$$B\left(a_{k_0(n)}^{(n)}, \frac{1}{n}\right) \subseteq B(y, \epsilon) \subseteq O_\beta$$

which is trivially true. But this is a contradiction which completes this part of the proof.
$((ii) \implies (i))$ *Suppose that any open cover of* X *has a finite subcover. Then we claim that* X *is complete, and that* X *is totally bounded. To show total boundedness, we let* $\epsilon > 0$ *and then*

$$X \subseteq \bigcap_{x \in X} B(x, \epsilon = U_x)$$

has a finite subcover. Thus, there exists $x_1, \ldots, x_{n(\epsilon)}$ *such that*

$$X \subseteq \bigcap_{k=1}^{n(\epsilon)} B(x_k, \epsilon)$$

is a finite ϵ*-net and* ϵ *is arbitrary, so total boundedness follows.*
To show completeness, we let

$$\cdots \subseteq B_k \subseteq \cdots \subseteq B_1$$

be a sequence of closed, nested spheres of radius $r_k \to 0$ *as* $k \to \infty$*. Now, suppose, for a contradiction, that*

$$\bigcap_{k=1}^{\infty} B_k = \emptyset$$

Now,

$$\bigcup_{k \in \mathbb{N}} (X \backslash B_k)$$

is an open cover of X*. It cannot have a finite subcover, otherwise* $B_j = \emptyset$ *for* $j > N$ *which is a contradiction that completes the final portion of the proof.* $\square$

Remark. Open cover property is usually taken as a definition of compactness for general topological spaces. To get a "sequential" definition for general topological spaces, one should generalize the notion of sequence to "nets."

THEOREM 45. *If X is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact.*
Proof.
Let

$$f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$$

Then, for any α, define $f^{-1}(V_\alpha) = U_\alpha$ which is open and $\{U_\alpha\}_{\alpha \in I}$ is an open cover of X which is compact. Then there exists a finite subcover

$$X \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$$

which implies that

$$f(X) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$$

is a finite subcover of $f(X)$ which yields our claim. $\square$

THEOREM 46. *If X is compact and $f : X \to Y$ is continuous, one-to-one and onto, then f^{-1} is also continuous and hence f is also a homeomorphism.*
Proof.
Define $Y = f(X)$ to be compact. Let $B \subseteq X$ be closed. Then B is compact since closed subsets of compact metric spaces are compact. Now, $f(B) = C$ is closed in Y which completes the proof. $\square$

If we let X, Y be compact metric spaces and if $C(X, Y)$ is the set of continuous functions from X to Y. Now, let $f, g \in C(X, Y)$ and define

$$d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$$

THEOREM 47. *Let $D \subseteq C(X, Y)$ where X, Y are compact. Then D is compact in $C(X, Y)$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$,*

$$d_X(x_1, x_2) < \delta \Longrightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

for every $f \in D$.
Proof. (Sketch)
Let $M_{X,Y}$ be the space of all mappings (not necessarily continuous) from X to Y. Define d_∞ as before. Now, $C(X, Y)$ is a closed subset of $M_{X,Y}$. Also, if $f_n : X \to Y$ converges uniformly with respect to d_∞ then the limit function is continuous.

THEOREM 48. *A closed subset Y of a compact set X is compact.*
Proof.
Let (U_α) be a cover of Y. $(U_\alpha) \cup (x \backslash Y)$ is an open cover of X which is compact. This implies that a finite subcover $U_{\alpha_1}, \ldots U_{\alpha_k}$, $(X \backslash Y)$ possibly of X. Hence, the subcover covers Y. $\square$

PROPOSITION 49. *Let X be a metric space. Then Y is sequentially compact in X if and only if $\overline{Y}$ is compact in itself.*

PROPOSITION 50. *Any function that is continuous on a compact metric space X is uniformly continuous.*
Proof.
Suppse, for a contradiction that there exists $\epsilon > 0$ such that there exists $(x_n), (x'_n)$ in X such that

$$d(x_n, x'_n) < \frac{1}{n} \Longrightarrow |f(x_n) - f(x'_n)| \geq \epsilon$$

Now, X is compact, so there exists $x_{n_k} \to y \in X$. Then $x'_{n_k} \to y$. f is continuous and so $f(x_{n_k}), f(x'_{n_k}) \to f(y)$. This contradicts our assumption and the proof is complete. $\square$

PROPOSITION 51. *If X is compact, then $f : X \to \mathbb{R}$ attains a least upper bound U and greatest lower bound L.*
Proof.
Suppose, for a contradiction, that U is not attained. Then, for each n, there exists $x_n \in X$ such that

$$U > f(x_n) \geq U - \frac{1}{n}$$

by sequential compactness, $x_{n_k} \to y \in X$. Now, by continuity, $f(x_{n_k}) \to f(y)$. Thus, $f(y) = U$ which contradicts our assumption and completes the proof for attaining the supremum. For the infimum, it is symmetric. We just replace f by $-f$. $\square$

THEOREM 52. *(General Arzela)*
Let $D \subseteq C(X,Y)$ where X, Y are compact metric spaces. Then D is sequentially compact in $C(X,Y)$ (which is equivalent to $\overline{D}$ is compact in $C(X,Y)$) if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$,

$$d(x_1, x_2) < \delta \Longrightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

for every $f \in D$.
Proof.
We remarked that it suffices to show that it suffices to show that D is totally bounded in $M(X,Y)$ with respect to

$$d_\infty(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$$

where $M(X,Y)$ is the space of all maps from X to Y (not necessarily continuous). Also, it was proven that $C(X,Y)$ is closed in $M(X,Y)$
To prove that D is totally bounded, we shall be approximating functions in D by piecewise constant (but discontinuous) functions.
Now, let $\epsilon > 0$, and in the definition of equicontinuity, choose $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \Longrightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

for all $f \in D$. Let $\{x_1, \ldots, x_n\}$ be a $(\delta/2)$-net in X. Then let

$$A_1 = B\left(x_1, \frac{\delta}{2}\right); \quad A_2 = B\left(x_2, \frac{\delta}{2}\right) \setminus A_1; \quad A_3 = B\left(x_3, \frac{\delta}{2}\right) \setminus (A_1 \cup A_2)$$

so that

$$A_k = B\left(x_k, \frac{\delta}{2}\right) \setminus (A_1 \cup \cdots \cup A_{k-1})$$

Now, $A_1 \cup \cdots \cup A_n = X$. The A_j are disjoint. Also, if $x_1, x_2 \in A_i$, then

$$d_X(x_1, x_2) < \delta$$

Now, if Y is compact, then there exists a finite $(\epsilon/2)$-net $\{y_1, \ldots, y_m\} \subseteq Y$. We shall approximate functions in D by the set of mappings that are constant on A_j's and take values in $\{y_1, \ldots, y_m\}$. This is a finite set and the number of functions is less than or equal to m^n.
We call this set Φ and we claim that for every $f \in D$, there exists $\varphi \in \Phi$ such that $d_\infty(f, \varphi) \leq 2\epsilon$.
To this end, for all $1 \leq i \leq n$, there exists $1 \leq j \leq m$ such that

$$d_Y(f(x_i), y_j) < \epsilon$$

Now, let $\varphi \in \Phi$ be defined by letting $\varphi(x_i) = y_j$, then

$$d_Y(f(x), \varphi(x)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), \varphi(x_i)) + d_Y(\varphi(x), \varphi(x_i))$$

The first two terms are less than ϵ and the last term is 0. Thus,

$$d_Y(f(x), \varphi(x)) < 2\epsilon$$

and the proof is complete. $\square$

EXAMPLE 17. *Let $f_n(x) = x^n$ and we wish to show that $\{f_n\} \subseteq C([0,1])$ is uniformly bounded and uniformly equicontinuous.*
Clearly, it is uniformly bounded by 1. For uniform equicontinuity, we have

$$f_n'(x) = nx^{n-1}$$

and so

$$f_n'(1) = n$$

We suppose for a contradiction that for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in X$,

$$|x - y| < \delta \Longrightarrow |f_n(x) - f_n(y)| < \epsilon$$

for each $n \in \mathbb{N}$. By Taylor's theorem, we get

$$|f_n(x) - f_n(y)| = |f_n'(x)| \cdot |x - y|$$

the rest is provided in a note online.

Chapter 5

Basic Point-Set Topology

Section 5.1

Product Topology

$$U_1 \times \cdots \times U_n \times \cdots$$

The basis of open sets are the sets where

(i) U_j's are open in X_j's

(ii) $U_j = X_j$ except for finitely many j's

Proposition 53. *Let*

$$f : Y \to X = \prod_{i=1}^{\infty} X_i$$

be a function. Then f is continuous (with respect to the product topology) if and only if

$$f(y) = [f_1(y), \cdots, f_n(y), \cdots]$$

where the $f_i(y)$ are continuous for each i.

Proof.

It suffices to show that f^{-1}(Basis element of the product topology) is open in Y. Let

$$U = U_1 \times U_n \times X_{n+1} \times X_{n+2} \times \cdots$$

and so

$$f^{-1}(U) = f_1^{-1}(U_1) \cap \cdots \cap f_n^{-1}(U_n) \cap Y \cap Y \cap \cdots = V$$

is open, as we disregard the intersections with Y and also since each $f_i^{-1}(U_i)$ is open. □

Example 18. *If we let*

$$f : \mathbb{R} \to \mathbb{R}^{\infty}$$

be such that

$$x \mapsto (x, \cdots, x)$$

then we can see that f is not continuous in the box topology.

Theorem 54. *(Easy Version Of Tikhov Theorem)*

Suppose that X_j is compact for each j, then

$$X = \prod_{i=1}^{\infty} X_i$$

with the product topology is also compact.

Proof.

Suppose that X has an open cover O that has no finite subcover. We first claim that there exists $x_1 \in X_1$ such that no basis set of the form

$$U_1 \times U_2 \times \cdots$$

is overed by finitely many open sets O. To show that this claim holds, we will suppose, for a contradiction that for any x_1, there exists U_1 containing x_1 such that

$$U_1(x_1) \times X_2 \times \cdots$$

is covered by finitely many sets in O. Then the sets

$$\{U(x_1) : x_1 \in X_1\}$$

are open covers of X. If X_1 is compact, then

$$X_1 \subseteq U_1(x_1) \cup \cdots \cup U_1(x_k)$$

and each $U_j \times X_2 \times \cdots \times X_n \times \cdots$ is covered by finitely many open sets in O. Thus

$$X_1 \times \cdots \times X_n \times \cdots$$

is covered by finitely many sets in O which contradicts our assumption proving the claim.
Now, by induction, there exists $X_2 \in X_2$ such that no basis set of the form

$$U_1 \times U_2 \times X_3 \times \cdots \times X_n \times \cdots$$

such that $(x_1, x_2) \in U_1 \times U_2$ can be covered by finitely many open sets in O. This is step 2 in the induction. This is proven using the compactness of X_2. Now, for step k, we have that for each $k \in \mathbb{N}$, there exists $x_k \in X_k$ such that no element of the form

$$U_1 \times \cdots \times U_k \times X_{k+1} \times \cdots$$

can be covered by finitely many open sets in O. Now, consider

$$\underline{x} = (x_1, \cdots, x_n, \cdots) \in X_1 \times \cdots \times X_k \times \cdots$$

and so there exists $v \in O$, such that $x \in V$. So there exists a basis element

$$\underline{U} = U_1 \times \cdots \times U_m \times X_{m+1} \times \cdots$$

which contradicts step m of the induction. $\square$

Suppose now that X is a metric space. Is the product topology metrizable? The answer is yes. We wish to put a metric on

$$\prod_{j=1}^{\infty} X_j$$

Step 1. Let d_j be the metric on X_j. Replace d_j by

$$\tilde{d}_j(x, y) = \frac{d_j(x, y)}{1 + d_j(x, y)} \leq 1$$

It is a fact that $\tilde{d}_j$ preserves topology on X_j.
Step 2. Let

$$\underline{x} = (x_1, \cdots, x_n, \cdots) \quad \underline{y} = (y_1, \cdots, y_n, \cdots)$$

and we say that

$$D(\underline{x}, \underline{y}) = \sum_{k=1}^{\infty} \frac{\tilde{d}_j(x_k, y_k)}{2^k}$$

and we're done!

Example 19. *We recall that the rational numbers can be completed to p-adic rational numbers, so*

$$\mathbb{Q} \to \mathbb{Q}_p = \left\{ \sum_{j=-m}^{\infty} a_j p^j : m \in \mathbb{N} \text{ finite}, 0 \leq a_j \leq p-1 \right\}$$

Section 5.2

Conectedness

Definition 42. *Let $A \subseteq X$ and we say that ∂A is the **Boundary** of A if ∂A is the set of points $x \in X$ such that $\exists x_n \in A$ such that $x_n \to x$ and $\exists y_n \in A^C$ such that $y_n \to x$ also.*

Proposition 55. *Let X be a metric space and let $A \subseteq X$. Then, ∂A is closed, and also $\partial(A \cup B) \subseteq \partial A \cup \partial B$. We can also find an example where $\partial(A \cup B) \neq \partial A \cup \partial B$.*
Proof.
Excercise.

Definition 43. *Let X be a topological space. We say that X is **Connected** if and only if we have for subsets $A, B \subseteq X$ either both open or both closed with $A \cap B = \emptyset$ that*

$$X = A \cup B \Longrightarrow \begin{cases} A = \emptyset, B = X & \text{or} \\ A = X, B = \emptyset \end{cases}$$

*If X is not connected, then X is called **Disconnected**.*

We make the remark that if $X = A \cup B$, then A, B are both closed and so.

Proposition 56. *X is connected if and only if we have that the only subsets that are both open and closed are $\emptyset$ and X.*

Example 20. *Any set with discrete topology with greater than 2 elements is disconnected.*

Proposition 57. *The interval $[a, b]$ is connected.*
Proof.
Suppose that $[a, b] = A \cup B$ where A, B are both open in $[a, b]$ and $A \cap B = \emptyset$ with $A, B \neq \emptyset$ (so, we're looking for a contradiction). Suppose that $a \in A$, then $[a, a + \epsilon) \subseteq A$ for some $\epsilon > 0$ which must happen since A is open. Let

$$C = \{c \in (a, b] : [a, c] \subseteq A\}$$

Clearly, $b \notin C$. Let $L = \sup C$, then either $L \in A$ or $L \in B$.
If $L \in A$ which is open, then we can find $\epsilon_1 > 0$ such that

$$(L - \epsilon_1, L + \epsilon_1) \subseteq A$$

Then

$$\left[a, L + \frac{\epsilon}{2}\right] \subseteq A$$

so L cannot be an upper bound which contradicts our assumption.
If $L \in B$, then we can find $\epsilon_2 > 0$ such that

$$(L - \epsilon_2, L + \epsilon_2) \subseteq B$$

and so L cannot be the least upper bound which is also a contradiction to our assumption. $\square$

DEFINITION 44. *X is called* ***Path Connected*** *if for each $x, y \in X$ there exists a continuous map $f : [a, b] \to X$ such that*

$$f(a) = x \quad f(b) = y$$

PROPOSITION 58. *If X is path connected, then X is connected.*
Proof.
Suppose for a contradiction that X is disconnected. Thus $X = A \cup B$ where A, B are open and $A \cap B = \emptyset$. Then, $U = f^{-1}(A)$ is open in $[a, b]$ and $V = f^{-1}(B)$ is open in $[a, b]$. Now, $U \cup V = [a, b]$ since $[a, b]$ is connected and we have a contradiction which completes the proof. $\square$

COROLLARY 59. *All open and half open intervals are connected.*

COROLLARY 60. *Convex sets are connected.*

DEFINITION 45. *If X is a linear space, then $A \subseteq X$ is* ***Convex*** *if for every $x, y \in A$, we have*

$$\{tx + (1 - t)y; t \in [0, 1]\} \subseteq A\}$$

That is, there is a line segment connecting every $x, y \in A$.

COROLLARY 61. *Any star like set B in $\mathbb{R}^n$ is connected.*

PROPOSITION 62. *Let $f : X \to Y$ be continuous and surjective. Then*

(i) If X is connected, then Y is connected.

(ii) If X is path connected, then Y is path connected.

Proof.

(i) Suppose for a contradiction that Y is disconnected, then $Y = A \cup B$ where A, B are open with $A \cap B = \emptyset$. Now, $U = f^{-1}(A)$ and $V = f^{-1}(B)$ are both open, with $U \cap V = \emptyset$ and also, $U \cup V = \emptyset$ and $U \cup V = X$.

(ii) Let $y_1, y_2 \in Y$ and let $x_1 \in f^{-1}(\{y_1\})$ and $x_2 \in f^{-1}(\{y_2\})$. We know that X is path connected, so there exists a continuous map $h : [a, b] \to X$ such that $h(a) = x_1$ and $h(b) = x_2$. Now, $f \circ h : [a, b] \to Y$ is also continuous since both f and h are continuous, so

$$(f \circ h)(a) = f(x_1) = y_1$$

$$(f \circ h)(b) = f(x_2) = y_2$$

so Y is path connected. $\square$

THEOREM 63 (Intermediate Value Theorem). *Let X be connected, and let $f : X \to \mathbb{R}$ be a continuous map and also, let $a < b$. If there exists $x_1 \in X$ such that $f(x_1) = a$ and there exists $x_2 \in X$ such that $f(x_2) = b$, then for any $c \in (a, b)$, there exists $y \in X$ such that $f(y) = c$.*
Proof.
Suppose for a contradiction that there exists $c \in (a, b)$ such that $f(y) \neq c$ for each $y \in X$. Then $f(X)$ cannot be connected. This is because

$$f(X) \subseteq (-\infty, c) \cup (c, +\infty)$$

and so if we let $U = f^{-1}((-\infty, c))$ and $V = f^{-1}((c, \infty))$, then $U \cap V = \emptyset$ and $U \cup V = X$. Also, $U, V \neq \emptyset$ with $x_1 \in U, x_2 \in V$ which explains the impossibility of connectedness in this case which contradicts our assumption. $\square$

Corollary 64. *There exists two antipodal points $x, -x$ on earth such that $T(x) = -T(-x)$.*
Proof.
Let $f : S^2 \to \mathbb{R}$ where $f(x) = T(x) - T(-x)$. If $f(y) = 0$, then we're done. Thus, we suppose for a contradiction that $f(y) = C \neq 0$, so $f(-y) = -C$. S^2 is path-connected which implies that it's continuous. Now, $0 \in (-c, c)$. By the intermediate value theorem. $\square$

Section 5.3

Connected Components/ Path Components

Suppose that X is not path connected and let $x \in X$, then we define

$$\begin{aligned} P(x) &= \{y \in X : \exists \text{ a continuous } x \to y \text{ path}\} \\ &= \{y \in X : \exists\, f : [a,b] \to X \ s.t. \ f(a) = x, f(b) = y\} \end{aligned}$$

Proposition 65. *$P(x)$ is path connected.* ***Proof.***
Follow that path from y_1 to x then from x to y_2.

Proposition 66. *If $x \sim y$ is defined to be the relation where $y \in P(x)$, then it is an equivalence relation.*
Proof.

1. *$x \sim x$*
2. *$x \sim y$ implies $y \sim x$*
3. *$x \sim y$ and $y \sim z$ implies $x \sim z$.*

Path components are equivalence classes of $\sim$. If $y \notin P(x)$, then $P(x) \cap P(y) = \emptyset$.

Proposition 67. *Let $A \subseteq X$ if A is connected, then $\overline{A}$ is also connected.*
Proof.
Suppose for a contradiction that A is connected but $\overline{A}$ is disconnected. Then $Cl(A) = B \cup C$ where both B and C are closed in $\overline{A}$ and hence also in X. Now, we know that a closed set B in $\overline{A}$ have the form $\tilde{B} \cap \overline{A}$ where $\tilde{B}$ is closed in X. So

$$A = (B \cap A) \cup (C \cap A)$$

where B and C are closed in X (by our remark above) and so if we let

$$B \cap A = B_1 \qquad C \cap A = C_1$$

and then

$$A = B_1 \cup C_1$$

where B_1, C_1 are both closed in A. A is connected, so one of B_1 and C_1 must be empty. Suppose that $C_1 = \emptyset$, then $B \cap A = A$ and thus $B \cap \overline{A} = \overline{A}$ and so $C = \emptyset$ which contradicts our assumption that $\overline{A}$ is disconnected, and completes the proof. $\square$

Lemma 68. *Let $A \subseteq X$ where A is both open and closed and let $C \subseteq X$ be connected. Then if $C \cap A \neq \emptyset$ then $C \subseteq A$.*
Proof.

Let A be both open and closed in X. Thus, $A \cap C$ is both oen and closed in C. C is connected, so if $C \neq \emptyset$, then we know that

$$A \cap C = \begin{cases} C \\ \emptyset \end{cases}$$

but by what we just said above, we're done and $A \cap C$ is forced to be equal to C. $\square$

PROPOSITION 69. *Let $(C_\alpha)_{\alpha \in I}$ be a family of connected subspaces of X. Suppose that for any $\alpha, \beta \in I$, we have $C_\alpha \cap C_\beta \neq \emptyset$. Then*

$$\bigcup_{\alpha \in I} C_\alpha$$

is connected.

Proof.

We use the lemma for this proof. The details are left to the reader as an excercise.

An application of this would be if C_α is the collection of all connected subsets $Y \subseteq X$ such that $x \in X$. By the above proposition,

$$\bigcup_{\alpha \in I} C_\alpha$$

is connected.

DEFINITION 46. *Let $x \in X$ and let $(C_\alpha)_{\alpha \in I}$ be the collection of all connected subsets $Y \subseteq X$ containing x. Then we say that*

$$C(x) = \bigcup_{\alpha \in I} C_\alpha$$

*is the **Connected Component** of x.*

PROPOSITION 70. *If $y \notin C(x)$ then $C(x) \cap C(y) = \emptyset$.*

PROPOSITION 71. *If each point in X has a neighbourhood that is path connected, then path components in X are connected components.*
$x \sim y$ if $y \in C(x)$ is an equivalence relation.

EXAMPLE 21. *Open subsets of $\mathbb{R}^n$. Open sets of a normed linear space.*

Chapter 6

Banach Space Techniques

Definition 47. *A complex vector space X is said to be **Normed** if there exists a function $||\cdot|| : X \to \mathbb{C}$ such that*

(i) $||x|| \geq 0 \quad \forall\, x \in X$

(ii) $||x|| = 0 \iff x = 0$

(iii) $||\alpha x|| = |\alpha|\, ||x|| \quad \forall\, x \in X, \alpha \in \mathbb{C}$

(iv) $||x + y|| \leq ||x|| + ||y|| \quad \forall\, x, y \in X$

Proposition 72. *The space (X, d) where $d(x, y) = ||x - y||$ always defines a metric space.*
Proof.
Simply check the four axioms of a distance. The only non-trivial part is the triangle inequality, so for any $x, y, z \in X$, we have

$$d(x, y) = ||x - y|| = ||x - z + z - y|| \leq ||x - z|| + ||z - y|| = d(x, z) + d(z, y) \qquad \square$$

Definition 48. *A normed vector space $(X, ||\cdot||)$ is said to be a **Banach Space** if it is complete with respect to the metric induced by $||\cdot||$.*

Example 22. *(a) Consider the space*

$$X = C^0([a, b]) = \{f : [a, b] \to \mathbb{C} : f \text{ is continuous}\}$$

with norm

$$||f||_\infty = \sup_{x\in[a,b]} |f(x)|$$

this is a Banach space.

(b) Consider

$$X = l_2 = \left\{ x : \mathbb{N} \to \mathbb{C} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with norm

$$||x|| = \sqrt{(x, x)}$$

and this space is a Hilbert space which implies that it is Banach.

Section 6.1

Linear Functionals

Definition 49. *A map $A : X \to Y$ between vector spaces X, Y is said to be **Linear** if*

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

for any $x, y \in X$ and $\alpha, \beta \in \mathbb{C}$.

Definition 50. *If a linear map $A : X \to Y$ between normed vector spaces X, Y has norm*

$$||A|| = \sup\{||Ax||_Y : x \in X, ||x||_x \leq 1\}$$

*and saitisfies $||A|| < \infty$, then we say that A is a **Bounded Linear Map**.*

Proposition 73. *If A is bounded, then the following are true.*

(i)

$$||A|| = \inf\{K > 0 : ||Ax||_Y \leq K||x||_X\} = \sup\{||Ax|| : ||x|| = 1\}$$

(ii)

$$||Ax||_Y \leq ||A|| \, ||x||_X$$

(iii) A maps $B_1^X(0) = \{x : ||x|| \leq 1\}$ into $B_{||A||}^Y(0) = \{y : ||y|| \leq ||A||\}$.

Definition 51. *If $Y = \mathbb{C}$, then the linear map $A : X \to \mathbb{C}$ is said to be a **Linear Functional** (not necessarily bounded).*

Theorem 74. *Let $A : X \to Y$ be linear and X, Y be normed linear spaces. The following are equivalent.*

(i) A is bounded.

(ii) A is continuous.

(iii) A is continuous at some $x_0 \in X$.

Proof.
(i) $\Rightarrow$ (ii)

$$||Ax_1 - Ax_2|| = ||A(x_1 - x_2)||_Y \leq ||A|| \, ||x_1 - x_2||_X$$

and continuity follows by taking $\epsilon > 0$ and

$$0 < \delta < \frac{\epsilon}{||A||}$$

(ii) $\Rightarrow$ (iii) *Trivial.*
(iii) $\Rightarrow$ (i) *Let A be continuous at $x_0 \in X$. Take $\epsilon > 0$, and there exists $\delta > 0$ such that for any $x \in X$,*

$$||x - x_0|| < \delta \Longrightarrow ||A(x - x_0)|| < \epsilon$$

Let $||h|| \leq \delta$ so that

$$||x_0 + h - x_0|| < \delta \Longrightarrow ||A(x_0 + h) - Ax_0|| = \frac{||Ah||}{\delta} < \frac{\epsilon}{\delta}$$

Then, if $||x|| \leq 1$, then

$$||Ax||| \leq \frac{\epsilon}{\delta} \Longrightarrow ||A|| \leq \frac{\epsilon}{\delta}$$

and then A is bounded. $\square$

Example 23. *Let $A : C^0([a, b]) \to C^0([a, b])$ and $K : [a, b]^2 \to \mathbb{C}$ be continuous, then*

$$(Af)(t) = \int_a^b K(t, s) f(s) ds$$

is bounded.

Theorem 75 (Baire's Category Theorem). *If (X, d) is a complete metric space, then the intersection of every countable collection of dense open subsets of X is dense in X. In particular, the intersection is nonempty.*

Proof.

Let $V_1, \ldots, v_n, \ldots \subseteq X$ be open and dense. So, for any i, $\overline{V}_i = X$ and V is open. Let $x_0 \in X$ and look at

$$B_X(x_0, \epsilon) = \{x \in X : d(x, x_0) < \epsilon\}$$

We want to show that

$$B_X(x_0, \epsilon) \cap \left(\bigcap_{n=1}^{\infty} V_n \right) \neq \emptyset$$

First, we notice that since V_1 is dense, we have that $B_X(x_0, \epsilon) \cap V_1 \neq \emptyset$ so there exists $x_1 \in B_X(x_0, \epsilon) \cap V_1$ and there exists $r_1 > 0$ such that

$$B_X(x_1, r_1) \subseteq B_X(x_0, \epsilon) \cap V_1$$

and continuing after n steps, $B_X(x_{n-1}, r_{n-1}) \cap V_n \neq \emptyset$ so there exists $x_n \in B_X(x_{n-1}, r_{n-1}) \cap V_n$ and

$$0 < r_n < \frac{1}{n}$$

so that

$$B_X(x_n, r_n) \subseteq B_X(x_{n-1}, r_{n-1}) \cap V_n$$

So this is in fact a sequence of nested balls $B_X(x_n, r_n) \subseteq \cdots \subseteq B_X(x_0, \epsilon)$ with the sequence $(x_n)_{n=1}^{\infty}$ and

$$d(x_{n+k}, x_{n+m}) \leq 2r_n \leq \frac{2}{n} \to 0$$

as $n \to \infty$ so this sequence is Cauchy. Thus, by completeness, there exists x^* such that

$$x_n \to x^*$$

as $n \to \infty$. Now, $x^* \in \overline{B_X(x_n, r_n)} \in V_n$ for every n and thus

$$x^* \in \bigcap_{n=1}^{\infty} V_n$$

and $x^* \in B_X(x_0, \epsilon)$, and finally

$$x^* \in B_X(x_0, \epsilon) \cap \left(\bigcap_{n=1}^{\infty} V_n \right) \neq \emptyset$$

and this completes the proof.

Corollary 76. *If (X, d) is complete then any countable intersection of G_δ supersets of X is again G_δ dense.*

Proof.

$$G_\delta = \bigcap_{i=1}^{\infty} U_i$$

for U_i open in X. If G_δ is dense, then so are the U_i and we have

$$\bigcap_{i=1}^{\infty} G_\delta^i = \bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty} U_i^j = \bigcap_{i,j=1}^{\infty} U_i^j$$

which by Baire's Category Thoeorem is dense and $E \subseteq X$ is nowhere dense if $\overline{E}$ contains no nonempty open subsets of X so $X \backslash E$ is open and dense. □

DEFINITION 52. *We say that the set F is **First Category** where*

$$F = \bigcap_{i=1}^{\infty} E_i$$

and E_i is nowhere dense if everything else is second countable.

THEOREM 77. *Let (X, D) be complete, then X is not first category.*

THEOREM 78 (Banach Steinhaus). *Let X be Banach, and let Y be a normed vector space and let $(A_\alpha)_{\alpha\in a}$ be a collection of bounded linear maps $A_\alpha : X \to Y$, then either*

- *There exists $M < \infty$ such that $\|A_\alpha\| \leq M$ for each $\alpha \in a$ **or***
- $$\sup_{\alpha\in a} \|A_\alpha x\| = \infty$$

  for all $x \in F$, where F is G_δ dense.

Proof.
Let $\varphi_\alpha(x) = \|A_\alpha x\|_Y$ and let

$$\varphi_\alpha = \sup_\alpha \varphi_\alpha(x) = \sup_\alpha \|A_\alpha x\|$$

and $\varphi, \varphi_\alpha : X \to \mathbb{R}$ where φ is a function and φ_α is continuous. Now, let

$$V_\alpha^n = \varphi_\alpha^{-1}(n, +\infty) = \{x \in X : \varphi_\alpha(x) > n\}$$

and let

$$V^n = \varphi^{-1}(n, +\infty) = \{x \in X : \varphi(x) > n\}$$

and we can see that

$$V^n = \bigcup_{\alpha\in a} V_\alpha^n$$

So

(i)

$$\begin{aligned} x \in \cup_\alpha V_\alpha^n \implies x \in V_{\alpha_0}^n \implies n < \varphi_{\alpha_0}(x) \leq \varphi(x) \\ \implies X \in V^n \end{aligned}$$

(ii)

$$x \in V^n \implies \sup_\alpha \varphi_\alpha(x) > n$$

assuming that for each α, $\varphi_\alpha(x) \leq n$, but then

$$\sup_\alpha \varphi_\alpha(x) \leq n$$

which is a contradiction and so there exists α_0 *such that* $\varphi_\alpha(x) > n$ *which implies that*

$$x \in V_{\alpha_0}^n \subseteq \bigcup_\alpha V_\alpha^n$$

and V^n *is then open because each* V_α^n *is open.*

Case 1. *There exists* $n_0 \in \mathbb{N}$ *such that* V^{n_0} *is not dense, and so there exists* $x_0 \in X$, $\delta > 0$ *such that*

$$\overline{B(x_0, \delta_0)} \cap \overline{V^{n_0}} = \emptyset$$

and if $\|x\|_X \leq \delta$ *then* $x_0 + x \notin V^{n_0}$ *but* $x_0 + x \in \overline{B(x_0, \delta_0)}$ *and so*

$$\varphi(x + x_0) \leq n_0$$

if and only if for each $\alpha \in a$, *we have*

$$\|A_\alpha x_0\| \leq n_0 \quad \|A_\alpha(x + x_0)\| \leq n_0$$

now let $x = (x + x_0) - x_0$ *such that* $\|x\| \leq \delta$, *then*

$$\|A_\alpha x\|_Y = \|A_\alpha(x + x_0) - A_\alpha x_0\| \leq \|A_\alpha(x + x_0)\| + \|A_\alpha x_0\| \leq 2n_0$$

and we can say that for any

$$\hat{x} = \frac{x}{\delta_0}$$

we have

$$\|\hat{x}\| \leq 1$$

and so

$$\|A_\alpha \hat{x}\| \leq \frac{2n_0}{\delta_0}$$

for every $\alpha \in a$, *and finally*

$$\sup_\alpha \|A_\alpha\| \leq \frac{2n_0}{\delta_0} = M$$

Case 2. *Every* V^n *is dense in* X. *By Baire's theorem,*

$$\bigcap_{n=1}^{\infty} V_i^n$$

is G_δ *dense in* X *so for every*

$$x \in \bigcap_{n=0}^{\infty} V^n$$

and $\varphi(x) = \infty$ *and we're done.* $\square$

This theorem can be interpreted as if X is Banach and Y is normed linear with our collection of A, then either there exists $B_Y(0, M) \subseteq Y$ such that for every $\alpha \in a$, we have

$$A_\alpha(B_X(0,1)) \subseteq B_Y(0, M)$$

or A_α maps F to $A_\alpha x$.
Now, before we continue, we first define for a Banach space X,

$$B_X(x_0, r) = \{x \in X : \|x - x_0\| < r\}$$

THEOREM 79 (Open Mapping Theorem). *Let X, Y be Banach spaces and let $A : X \to Y$ be a bounded linear map such that $A(X) = Y$ (i.e. A is onto), then there exists $\delta > 0$ such that*

$$B_Y(0, \delta) \subseteq A(B_X(0, 1))$$

An alternative way of stating this theorem is: For each y with $\|y\| < \delta$, there exists an x such that $\|x\| < 1$ such that $Ax = y$.

Proof.

Given $y \in Y$, there exists $x \in X$ such that $Ax = y$. If $\|x\| < k$, then it follows that $y \in A(kB_X(0, 1))$. Thus, Y is the union of the sets $A(kB_X(0, 1))$ for $k = 1, 2, \ldots$. Now, since Y is complete, the Baire Category Theorem implies that there exists a nonempty open set W in the closure of some $A(kB_X(0, 1))$. This means that every point of W is the limit of a sequence Ax_n, where $x_n \in kB_X(0, 1)$. Now we fix k and W.

Choose $y_0 \in W$ an choose $\eta > 0$ so that $y_0 + y \in W$ for any $\|y\| < \eta$. For any such y, there exist sequences $(x'_n), (x''_n)$ in $kB_X(0, 1)$ such that

$$Ax'_n \to y_0 \qquad Ax''_n \to y_0 + y$$

as $n \to \infty$. Setting $x_n = x''_n - x'_n$, we get that $\|x_n\| < 2k$ and thus $Ax_n \to y$ as $n \to \infty$. Since this holds for each y with $\|y\| < \eta$, we get from the linearity of A that for every $y \in Y$ and for all $\epsilon > 0$ there exists an $x \in X$ such that

$$(*) \qquad \|x\| \leq \delta^{-1}\|y\| \quad \& \quad \|y - Ax\| < \epsilon$$

when we simply put $\delta = \frac{\eta}{2k}$.

We're almost there, we just need $\epsilon = 0$.

First, take $y \in \delta B_Y(0, 1)$ and let $\epsilon > 0$. By $(*)$, we have that there exists x_1 with $\|x_1\| < 1$ and

$$\|y - Ax_1\| < \frac{1}{2}\delta\epsilon$$

Now, suppose that $x_1, \ldots, x_n$ are chosen so that

$$\|y - Ax_1 - \cdots - Ax_n\| < 2^{-n}\delta\epsilon$$

then using $(*)$ with y replaced by the vector on the left hand side of the above inequality, we obtain x_{n+1} so that the above holds with $n + 1$ in place of n, and

$$\|x_{n+1}\| < 2^{-n}\epsilon$$

for $n = 1, 2, \ldots$. Now, if we set $s_n = x_1 + \ldots + x_n$, then we get that (s_n) is a Cauchy sequence in X, and thus by completeness, there exists $x \in X$ such that $s_n \to x$. Then since $\|x\| < 1$, we use the above and get that $\|x\| < 1 + \epsilon$, and since A is continuous, $As_n \to Ax$. Now, by $(*)$, $As_n \to y$ and thus $Ax = y$. We have now that

$$\delta B_Y(0, 1) \subseteq A((1 + \epsilon)B_X(0, 1))$$

or

$$A(B_X(0, 1)) \subseteq \frac{1}{1 + \epsilon}\delta B_Y(0, 1)$$

for every $\epsilon > 0$. The union of the sets on the left, taken over each $\epsilon > 0$ is precisely $\delta B_Y(0, 1)$ which completes the proof. $\square$

THEOREM 80. *Let X and Y be Banach spaces and let A be a bounded linear functional from $X \to Y$ which is one to one, then there exists $\delta > 0$ such that for each $x \in X$*

$$\|Ax\| \geq \delta\|x\|$$

That is, $A^{-1} : Y \to X$ is also a bounded linear functional.

Proof.

If we choose δ as we chose in the Open Mapping Theorem, then the conclusion of that theorem, combined with the fact that A is now one to one shows that

$$\|Ax\| < \delta \Longrightarrow \|x\| < 1$$

Thus, $\|x\| \geq 1$ implies that $\|Ax\| \geq \delta$ and so $\|Ax\| \geq \delta\|x\|$, as needed.
The functional A^{-1} is defined on Y by the requirement that $A^{-1}y = x$ if $y = Ax$. A trivial verification then shows that A^{-1} is linear and

$$\|A^{-1}\| \leq \frac{1}{\delta}$$

follows from the fact that $\|Ax\| \geq \delta\|x\|$. $\square$

Definition 53. *A **Linear Manifold** L in a Banach space X is a set such that for any $x, y \in L$ and for any $\alpha, \beta \in \mathbb{R}$, we have*

$$\alpha x + \beta y \in L$$

*A linear manifold L is called a **Subspace** if and only if it is a closed subset of X.*

Example 24. *Consider $\mathbb{R}^\infty = X$. Let*

$$L = \{(x_1, \ldots, x_n, \ldots) : x_k = 0 \;\forall\; k > N\}$$

Then, L is a linear submanifold of X. L is not closed. If we have

$$x_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{2^n}, 0, 0, \ldots\right) \to x = \left(1, \frac{1}{2}, \ldots, \frac{1}{2^n}, \frac{1}{2^{n+1}}, \ldots\right)$$

and $x \notin L$.

Proposition 81. *Let X be Banach with $z_1, \ldots, z_n, \ldots \in X$. Consider*

$$M = \left\{\sum_j \alpha_j z_j : \alpha_j \neq 0 \text{ only for finitely many } j\right\}$$

then M is a linear submanifold of X. Let $\overline{M}$ be the closure of M. Then $\overline{M}$ is a linear subspace of X.
Proof.
First, we note that $\overline{M}$ is closed. We want to prove that M is a linear submanifold of X. Let $x, y \in \overline{M}$. Fix $\epsilon > 0$, then there exists $x_1, y_1 \in M$ such that

$$\|x_1 - x\| < \epsilon \quad \|y_1 - y\| < \epsilon$$

Let $\alpha, \beta \in \mathbb{R}$. and we examine

$$\begin{aligned}
\|\alpha x_1 + \beta y_1 - \alpha x_1 - \beta y_1\| &\leq \|\alpha x - \alpha x_1\| + \|\beta y - \beta y_1\| \\
&\leq |\alpha| \cdot \|x - x_1\| + |\beta| \cdot \|y - y_1\| \\
&\leq (|\alpha| + |\beta|)\epsilon
\end{aligned}$$

and ϵ is arbitrary, and $(\alpha x_1 + \beta y_1) \in M$ is close to $\alpha x + \beta y$ which implies that $\alpha x + \beta y \in \overline{M}$ as required.
In the statement of this proposition, $\overline{M}$ is a subspace generated by $z_1, \ldots, z_n, \ldots$). $\square$

Convex Sets

Definition 54. *Let $A \subseteq X$ be a linear space. A is said to be a **Convex Set** if for every $x, y \in A$, we have that*

$$\alpha x + \beta y \in A$$

where $\alpha, \beta \in [0,1]$ and $\alpha + \beta = 1$.
*We say that A is a **Convex Body** if A is convex and A has an interior point (that is, it contains some ball).*

Example 25. *Let $X = l_2$. Let*

$$\Phi = \left\{ (\xi_1, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} n^2 \xi_n^2 < \infty \right\}$$

we claim that Φ is convex but does not contain any ball in l_2 (that is, it's not a convex body).
Proof.
Let $\xi = (\xi_1, \ldots, \xi_n, \ldots) \in \Phi$, let $\eta = (\eta_1, \ldots, \eta_n, \ldots) \in \Phi$. Let $t \in (0,1)$. We have to show that

$$t\xi + (1-t)\eta \in \Phi$$

so

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2 (t\xi_n + (1-t)\eta_n)^2 &= -t^2 \sum_{n=1}^{\infty} n^2 \xi_n^2 + 2t(1-t) \sum_{n=1}^{\infty} \xi_n \eta_n + (1-t)^2 \sum_{n=1}^{\infty} n^2 \eta_n^2 \\
&\leq t^2 \sum_{n=1}^{\infty} n^2 \xi_n^2 + 2t(1-t) \left(\sum_{n=1}^{\infty} n^2 \xi_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} n^2 \eta_n^2 \right)^{1/2} \\
&\quad + (1-t)^2 \sum_{n=1}^{\infty} n^2 \eta_n^2 \\
&= \left[t \left(\sum_{n=1}^{\infty} n^2 \xi_n^2 \right)^{1/2} + (1-t) \left(\sum_{n=1}^{\infty} n^2 \eta_n^2 \right)^{1/2} \right]^2 \\
&\leq [t \cdot 1 + (1-t) \cdot 1]^2 = 1
\end{aligned}$$

and thus we have that Φ is convex.
Now we suppose for a contradiction that Φ contains some ball. Φ is symmetric (that is, if $\xi \in \Phi$, then $-\xi \in \Phi$). Φ will contain all

$$\{ z = tx + (1-t)y : x \in B_1, y \in -B_1 \}$$

It is left as an excercise to show that if the radius of B_1 is r, then Φ will contain $\overline{B}(0,r)$ and so Φ should contain a segment of every line passing through 0. Then let

$$l = t \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots \right)$$

Then $l \cap \Phi = \{0\}$.
Suppose that $A \subseteq X$, where A is convex, symmetric. Suppose further that

$$B(x,r) \subseteq A$$

Then

$$B(0,r) \subseteq A$$

To prove this, we first notice that

$$B(x,r) = \{x + y : \|y\| \le r\}$$
$$B(-x,r) = \{-x + y : \|y\| \le r\}$$

So, let $y \in X$ such that $\|y\| \le r$ and we write

$$y = \frac{1}{2}[(x+y) + (-x+y)]$$

where the first term is in $B(x,r)$ and the second term is in $B(-x,r)$. Now we claim that if we take a line l in the direction

$$\left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)$$

Then

$$l \cap \left\{ x = (x_1, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} n^2 x^2 < \infty \right\} = (0, \ldots, 0, \ldots)$$

To this end, suppose that $A \cap l \neq 0$, then there exists $t \neq 0$ such that

$$z = \left(t, \frac{t}{2}, \ldots, \frac{t}{n}, \ldots\right) \in A$$

and consider

$$\sum_{n=1}^{\infty} n^2 \frac{t^2}{n^2} = \sum_{n=1}^{\infty} t^2 = \infty$$

and so $z \notin A$ which is a contradiction and which yields our claim.

Lemma 82. *If $Im(\alpha_j) > 0$, $Im(z) < 0$, we have*

$$\sum_{j=1}^{n} \frac{1}{z - \alpha_j} \neq 0$$

Theorem 83 (Gauss-Lucas). *Let $P(z)$ be a complex polynomial defined as*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

then

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

*and the roots of $P'(z)$ lie inside a **Convex Hull** of the roots of $P(z)$.*

Proof.

We first note that

$$CH(z_1, \ldots, z_n) = \bigcap_{B_\alpha \ half-plane_{z_1,\ldots,z_n \in B_\alpha}} B_\alpha$$

WLOG, assume that $P(z)$ has simple roots

$$P(z) = (z - \alpha_1) \cdots (z - \alpha_n)$$

then

$$\frac{P'(z)}{P(z)} = \frac{(z-\alpha_2)\cdots(z-\alpha_n)}{P(z)} + \cdots + \frac{(z-\alpha_1)\cdots(z-\alpha_{n-1})}{P(z)}$$
$$= \frac{1}{z-\alpha_1} + \cdots + \frac{1}{z-\alpha_n}$$

by the extended product rule applied to the factored polynomial $P(z)$. Now, the set of roots of $P'(z)$ is defined as

$$\left\{z : \frac{1}{z-\alpha_1} + \cdots + \frac{1}{z-\alpha_n} = 0\right\}$$

and applying the lemma yields our result.

THEOREM 84. *If A is convex, then $\overline{A}$ is also convex.*
Proof.
Let $x, y \in \overline{A}$. For every $\epsilon > 0$, there exists $x_1, y_1 \in A$ such that

$$\|x - x_1\| \leq \epsilon \qquad \|y - y_1\| \leq \epsilon$$

Now, let $t \in (0,1)$ and we want that

$$tx + (1-t)y \in \overline{A}$$

We know that

$$tx_1 + (1-t)y_1 \in A$$

and

$$\begin{aligned}\|tx + (1-t)y - tx_1 - (1-t)y_1\| &\leq \|tx - tx_1\| + \|(1-t)y - (1-t)y_1\| \\ &\leq t\|x - x_1\| + (1-t)\|y - y_1\| \\ &\leq t\epsilon + (1-t)\epsilon = \epsilon\end{aligned}$$

as required.

THEOREM 85. *Suppose that M_α is convex for each $\alpha \in I$, then*

$$\bigcap_{\alpha \in I} M_\alpha$$

is also convex.
Proof.
If

$$x, y \in \bigcap_{\alpha \in I} M_\alpha$$

then, $x, y \in M_\alpha$ for all $\alpha \in I$. Then $tx + (1-t)y \in M_\alpha$ for $t \in (0,1)$. Thus

$$[tx + (1-t)y] \in \bigcap_{\alpha \in I} M_\alpha$$

Now, if $x_1, \ldots, x_{n+1}$ are in general position, then $CH(x_1, \ldots, x_{n+1})$ is a simplex with vertices at $x_1, \ldots, x_{n+1}$. If no 3 points lie on the same straight line, then no 4 points lie in the same plane, and so on. If x_j is not in the subspace containing all $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}$.

We note that if $A : X \to \mathbb{R}$ is a bounded, continuous linear functional, then the norm of A is defined by

$$\|A\| = \sup_{x \in X : \|x\| = 1} |Ax|$$

EXAMPLE 26. *Consider $f \in C[a,b]$, and let*

$$Af = \int_a^b f(x)dx$$

is a linear functional. To compute the norm, we let $\|f\| \leq 1$ *so that*

$$\sup_x |f(x)| \leq 1$$

then

$$\left| \int_a^b f(x)dx \right| \leq \|f\| \cdot |b-a|$$

$$\implies \|A\| \leq |b-a|$$

so if

$$f = \chi_{[a,b]} \Rightarrow \|f\| = 1$$

and thus

$$\int_a^b 1dx = b-a$$

$$\implies \|A\| = |b-a|$$

Now consider $g \in C[a,b]$*, and define*

$$A_1 f = \int_a^b f(x)g(x)dx$$

and let

$$\sup_{x\in[a,b]} |g(x)| = M$$

then

$$\left| \int_a^b f(x)g(x)dx \right| \leq \int_a^b |f(x)| \cdot |g(x)|dx \leq \|f\| \int_a^b |g(x)|dx$$

and so

$$\|A_1\| \leq \int_a^b |g(x)|dx$$

One can show that

$$\|A_1\| = \int_a^b |g(x)|dx$$

Proof. (Sketch)
If $g(x) \geq 0$*, then this is easy. Just take* $f \equiv 1$ *on* $[a,b]$*. Then*

$$A_1 f = \int_a^b 1 \cdot g(x)dx = \int_a^b |g(x)|dx$$

But if $g(x)$ *crosses zero at some point in* $[a,b]$*, then the idea is to take a piecewise continuous function* f *defined by*

$$f(x) = \begin{cases} 1 & g \geq 0 \\ -1 & g < 0 \end{cases}$$

and approximate $sgn(g(x))$ *by continous functions.*

Kernel

Definition 55. *The **Kernel** of a continuous linear functional A is defined as*

$$ker(A) = \{x \in X : Ax = 0\}$$

Proposition 86. *Let $L = ker(A)$, then L is a linear submanifold (closed linear subspace) of X.*

Proof.

Suppose that $x_1, x_2 \in L$. Then

$$A(t_1x_1 + t_2x_2) = t_1Ax_1 + t_2Ax_2 = 0$$

If $x_j \in L$, and $x_j \to y$, then $0 = Ax_j \to A_y = 0$ and so

$$y \in L$$

and we're done. $\square$

Definition 56. *Let L be a linear subspace of a Banach space X. Then we say that L has **Index** k if and only if*

(i) There exists k linearly independent vectors $x_1, \ldots, x_k \in X$ such that for every $x \in X$, there exists $t_1, \ldots, t_k \in \mathbb{R}$ and $y \in L$ such that

$$(*) \qquad x = y + t_1x_1 + \cdots + t_kx_k$$

(ii) There is no set of $(k-1)$ elements $\tilde{x}_1, \ldots, \tilde{x}_{k-1}$ such that $(*)$ holds.

Theorem 87. *Let $A \neq 0$ be a continuous linear functional on X. Then $L(A)$ (i.e. the kernel of A) has index $k = 1$. That is, there exists $x_0 \notin L$ such that for any $y \in X$, there exists $\lambda \in \mathbb{R}$ and $x \in L$ such that*

$$y = x + \lambda x_0$$

Proof.

Since $A \neq 0$, there exists $x_0 \in X$ such that $Ax_0 \neq 0$. Let $y \in X$, let

$$\lambda = \frac{Ay}{Ax_0}$$

Now we want $x \in ker(A) = L$ so let

$$x = y - x_0\frac{Ay}{Ax_0}$$

$$\implies Ax = Ay - \frac{Ay}{Ax_0}(Ax_0) = 0$$

and so $x \in ker(A)$. Now we claim that if x_0 is fixed, then there is only one way to write $y = \lambda x_0 + x$. To this end, we suppose for a contradiction that

$$y = \lambda_1x_0 + x_1$$

Now, if $\lambda = \lambda_1$, then $x = x_1$, so $\lambda \neq \lambda_1$, so

$$(\lambda_1 - \lambda)x_0 = x - x_1$$

$$\implies x_0 = \frac{x - x_1}{\lambda_1 - \lambda}$$

now $x, x_1 \in L$ and so $x - x_1 \in L$ by linearity, and also

$$\frac{x - x_1}{\lambda_1 - \lambda} \in L$$

but we assumed that $x_0 \notin L$. This contradiction yields the result.
Conversely, if L is a subspace of X of index $k = 1$, then there exists a linear functional A such that $L = ker(A)$. Consider the set

$$L_1 = \{x \in X : Ax = 1\}$$

If $x_0 \in X$ such that $Ax_0 = 1$ then

$$L_1 = x_0 + ker(A)$$

so if $Ax_0 = A\tilde{x}_0 = 1$, then

$$A(x_0 - \tilde{x}_0) = 0$$

since both $x_0, \tilde{x}_0 \in ker(A)$. Now, we want to compute

$$d(L_1, 0) = \inf\{\|x\| : Ax = 1\}$$

We claim that $d(L_1, 0) = 1/\|A\|$. To show this, we suppose that $x \in L_1$ so that

$$|Ax| = 1 \leq \|A\| \cdot \|x\|$$

$$\implies \|x\| \geq \frac{1}{\|A\|}$$

and so

$$d \geq \frac{1}{\|A\|}$$

and the other direction follows by the definition of $\|A\|$.

Section 6.5

Conjugate Space

Suppose that X is a normed linear space and that A_1, A_2 are continuous functionals on X, then so is $t_1 A_1 + t_2 A_2$ for $t_j \in \mathbb{R}$.
$\|A\|$ defines a distance on the space X^* of all continuous linear functionals on X and we say that X^* is **Conjugate Space**. We also note that

$$\|A_1 + A_2\| \leq \|A_1\| + \|A_2\|$$

Theorem 88. *X^* is always complete whether X is complete or not.*
Proof.
Let $A_n : X \to \mathbb{R}$ be a cauchy sequence of linear functionals. Consider a sequence of real numbers $(A_n x)$. This is a Cauchy sequence in $\mathbb{R}$. Now, let $x \in X$ and so

$$|A_n x - A_m x| \leq \|A_n - A_m\|_{op} \cdot \|x\|$$

where the norm of the difference fo the functionals tends to zero as $m, n \to \infty$ and where the norm of x is fixed. Thus

$$A_n x \to Bx$$

as $n \to \infty$. *Now, since* $A_n x \to Bx$, $A_n y \to By$, *we have*

$$A_n(t_1 x + t_2 y) \to t_1 Bx + t_2 By$$

Let N *be such that*

$$\|A_n - A_{n+p}\| < 1$$

for $n \geq N$, $p \geq 0$. *Therefore,*

$$\|A_{n+p}\| \leq \|A_n\| + 1$$

So,

$$|A_{n+p} x| \leq (\|A_n\| + 1)\|x\|$$

and as $p \to \infty$, *we get*

$$|Bx| \leq (\|A_n\| + 1)\|x\|$$

and so B *is bounded since*

$$\|B\|_{op} = \sup_{\|x\|=1} \frac{|Bx|}{\|x\|} \leq (\|A_n\| + 1)\|x\| < \infty$$

It remains now to show that $\|A_n - B\|_{op} \to 0$ *as* $n \to \infty$. *To this end, let* $\epsilon > 0$, *then there exists* $x = x_\epsilon \in X$ *such that*

$$\begin{aligned} \|A_n - B\| &\leq \frac{|A_n x_\epsilon - B x_\epsilon|}{\|x_\epsilon\|} + \frac{\epsilon}{2} \\ &= \left| A_n \left(\frac{x_\epsilon}{\|x_\epsilon\|} \right) - B \left(\frac{x_\epsilon}{\|x_\epsilon\|} \right) \right| + \frac{\epsilon}{2} \end{aligned}$$

and we let

$$y = \frac{x_\epsilon}{\|x_\epsilon\|}$$

We know that

$$By = \lim_{n \to \infty} A_n y$$

So, there exists $n_0 = n_0(\epsilon)$ *such that for any* $n > n_0$, *we have*

$$|A_n y - By| < \frac{\epsilon}{2}$$

Thus

$$\|A_n - B\|_{op} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for any $n > n_0$. *Thus,* X^* *is in fact complete.*

Remark. We can replace $A_n : X \to \mathbb{R}$ by $T_n : X \to Y$, where Y is a linear normed space and complete. Then it follows that X^* is Banach. So, $A_n x = y_n \in Y$ will be a Cauchy sequence and Y is Complete, so $y_n \to z = Bx$, and the rest of the proof is virtually identical.
We have thus shown that the space of all bounded linear operators $T : X \to Y$ where Y is Banach is complete and moreover it is Banach.

Example 27. *Look at the space of sequences*

$$Y = \{x = (x_1, \ldots, x_n, \ldots) : x_n \to 0 \text{ as } n \to \infty\}$$

Now, let $a = (a_1, \ldots, a_n, \ldots) \in l_1$ *have*

$$\sum_{j=1}^{\infty} |a_j| < \infty$$

and let

$$Ax = \sum_{j=1}^{\infty} a_j x_j$$

we know that $|x_j| < \epsilon$ *for* $j \geq N$ *and so*

$$\sum_{j=N}^{\infty} |a_j x_j| \leq \epsilon \sum_{j=N}^{\infty} |a_j| \to 0$$

as $N \to \infty$.

Proposition 89.

$$\|A\| = \|a\|_1 = \sum_{j=1}^{\infty} |a_j|$$

Proof.
Let $Y \subseteq l_\infty$ *and let* $x \in Y$. *Now,*

$$\|x\| = \sup_{j=1}^{\infty} |x_j|$$

and so let $\|x_j\| < 1$. *Then*

$$|Ax| = \left| \sum_{j=1}^{\infty} a_j x_j \right| \leq \sup_{j=1}^{\infty} |x_j| \sum_{j=1}^{\infty} |a_j| = \|a_1\|_1$$

and so

$$\|A\|_{op} \leq \|a\|_1$$

For the converse, fix $\epsilon > 0$ *and suppose that*

$$\sum_{j=1}^{N} |a_j| \geq \|a\|_1 - \epsilon$$

Then, let

$$x_j = \begin{cases} 1 & a_j \geq 0, j \leq N \\ -1 & a_j < 0, j \leq N \\ 0 & j > N \end{cases}$$

and let $x = (x_1, \ldots, x_N, 0, 0, \ldots)$. *Then* $\|x\| = 1$ *and we have*

$$\sum_{j=1}^{\infty} a_j x_j = \sum_{j=1}^{N} |a_j| \geq \|a\| - \epsilon$$

as required. □

Section 6.6

Linear Functionals Revisited

Proposition 90. *Let*

$$C = \{x = (x_1, \ldots, x_n, \ldots) : x_n \to 0\}$$

and define the linear functional

$$Ax = \sum_{j=1}^{\infty} a_j x_j$$

where

$$\sum_{j=1}^{\infty} |a_j| < \infty$$

Then, $C^* = l_1$.
Proof.
Let $e_j = (0, \ldots, 0, 1, 0, \ldots) \in C$ *where there is a 1 at the* j^{th} *index and zeros everywhere else. Now, let* A *be a bounded linear functional on* C*, then*

$$Ae_j = a_j \qquad \forall\ a_j$$

If $x = (x_1, \ldots, x_n, 0, \ldots, 0, \ldots)$*, then we have that*

$$x = x_1 e_1 + \cdots + x_n e_n$$

so that

$$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Now we claim that

$$\sum_{j=1}^{\infty} |a_j| < \infty$$

To this end, we suppose, for a contradiction that

$$\sum_{j=1}^{\infty} |a_j| = \infty$$

but that $\|A\|_{op} < M$ *where* $M > 0$*. Thus, there exists* $N \in \mathbb{N}$ *such that*

$$\sum_{j=1}^{N} |a_j| > M$$

Now let

$$x_j = \begin{cases} 1 & j < N, a_j > 0 \\ -1 & j \leq N, a_j < 0 \\ 0 & j > N, \|x\|_\infty = 1 \end{cases}$$

then

$$Ax = \sum_{j=1}^{N} a_j \cdot sgn(a_j) = \sum_{j=1}^{N} |a_j| > M$$

but then $M > M$ *which contradicts our assumption and completes the proof.* $\square$

Example 28. *Consider* l_2 *and let* $x = (x_1, \ldots, x_n, \ldots)$ *such that*

$$\sum_{j=1} |x_j|^2$$

and let $a = (a_1, \ldots, a_n, \ldots)$ *such that*

$$\sum_{j=1} |a_j|^2$$

and define the linear functional

$$Ax = \sum_{j=1}^{\infty} a_j x_j$$

and we claim that $Ax < \infty$. *To this end we write*

$$|(a,x)| = \left| \sum_{j=1}^{\infty} a_j x_j \right| \leq \|a\|_2 \|x\|_2$$

so that

$$\|A\|_{op} = \sup_{\|x\| \neq 0} \frac{|Ax|}{\|x\|} = (a,x) \leq \|a\|_2 = (a_1^2 + \cdots + a_n^2 + \cdots)^{\frac{1}{2}}$$

now take $x = a$ *so that*

$$\frac{Ax}{\|x\|} = \frac{(a,a)}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|$$

and thus

$$\|A\|_{op} = \|a\|$$

Every bounded linear functional on l_2 *is obtained like this.*

Proposition 91. $(l_2)^* = l_2$.

Proof.

The proof for this is similar to the proof of the above proposition. Let $x = (x_1, \ldots, x_n, \ldots) \in l_p$ *so that*

$$\sum_{j=1}^{\infty} |x_j|^p < \infty$$

then let $a = (a_1, \ldots, a_n, \ldots) \in l_q$ *where*

$$\frac{1}{p} + \frac{1}{q} = 1$$

So,

$$Ax = \sum_{j=1}^{\infty} a_j x_j \leq \|a\|_q \|x\|_p$$

by Hölder's inequality. Thus,

$$\|A\|_{op} = \|a\|_q$$

and all bounded linear operators on $l_p, p \in (1, \infty)$ *will have this form and thus,*

$$(l_p)^* = l_q$$

for all $p, q \in (1, \infty)$ *such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

and we have proven something stronger than what we set out to prove! Bonus! □

Example 29. *Let* $x = (x_1, \ldots, x_n, \ldots) \in l_1$ *so that*

$$\sum_{j=1}^{\infty} |x_j| < \infty$$

and let $a = (a_1, \dots, a_n, \dots) \in l_\infty$ *so that*

$$\sup_{j=1}^{\infty} |a_j| \le T < \infty$$

Then

$$\sum_{j=1}^{\infty} a_j x_j < \infty$$

and so

$$\sum_{j=1}^{\infty} |a_j| < \infty$$

and thus

$$\|A\|_{op} = \sup_{j=1}^{\infty} |a_j|$$

and thus $(l_1)^* = l_\infty$.

We now know that $C^* = l_1$ and that $(l_1)^* = l\infty$, however $(l_\infty)^* = A$ where A is the space of horrible nightmares. The main point here though, is that C, l_1, l_∞ are not reflexive.

THEOREM 92. $X \subseteq (X^*)^*$, *and* $X \cong \{$ *a linear subspace of* $(X^*)^*\}$.
Proof (Idea).
Let A *be a bounded linear functional on* X *and let* $z \in X$. *Now, define* $l_z(A) = Az$ *(*δ *function at* z*). If* $A_1, A_2 \in X^*$ *then*

$$l_z(t_1A_1 + t_2A_2) = t_1A_1z + t_2A_2z = t_1l_z(A) + t_2l_z(A)$$

Now, if $\|A\| < 1$, *then*

$$|Az| \le \|A\|_{op} \cdot \|z\|_X$$

and thus

$$\|l_z\| \le \|z\|_X$$

and now we claim that $\|l_{op,X^{**}} = \|z\|_X$ *which follows directly from the Hahn-Banach Theorem.*

SECTION 6.7

BERNSTEIN POLYNOMIALS

LEMMA 93. *We claim that*

$$[x + (1-y)]^n = \sum_{k=0}^{n} x^k \binom{n}{k} (1-y)^{n-k}$$

Proof.
First,

$$\left(x\frac{d}{dx}\right): \quad nx[x + (1-y)]^{n-1} = \sum_{k=0}^{n} kx^k \binom{n}{k} (1-y)^{n-k}$$

$$x^2\left(\frac{d}{dx}\right)^2: \quad n(n-1)x^2[x + (1-y)]^{n-2} = \sum_{k=0}^{n} k(k-1) \binom{n}{k} (1-y)^{n-k}$$

Now evaluate both at $y = x$ *so that*

$$1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

$$nx = \sum_{k=0}^{n} kx^k \binom{n}{k} (1-x)^{n-k}$$

$$n(n-1)x^2 = \sum_{k=0}^{n} k(k-1)x^k \binom{n}{k} (1-x)^{n-k}$$

Now, adding the above three identities together yields our result.

Theorem 94 (Bernstein Approximation Theorem). *Let $f \in C([0,1])$ and define the n^{th} Bernstein Polynomial of f by*

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}$$

then

$$\sup_{x \in [0,1]} |f(x) - B_n(f;x)| \to 0$$

as $n \to \infty$.

Proof.

We consider

$$\begin{aligned}
&\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} \left(x^2 - \frac{2k}{n}x + \frac{k^2-k}{n^2} + \frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \\
&= x^2 \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} - \frac{2x}{n} \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k} \\
&\qquad + \frac{1}{n^2} \sum_{k=0}^{n} k(k-1) \binom{n}{k} x^k (1-x)^{n-k} + \frac{1}{n^2} \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k} \\
&= x^2 \cdot 1 - \frac{2x}{n} \cdot nx + \frac{1}{n^2} n(n-1)x^2 + \frac{1}{n^2} \\
&= \frac{x(1-x)}{n} = \frac{x-x^2}{n}
\end{aligned}$$

by applying the three identities from Lemma 88. Now, let $\delta > 0$ and fix $x \in [0,1]$ and look at

$$\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \geq \sum_{k:|x-k/n|>\delta} \delta^2 \binom{n}{k} x^k (1-x)^{n-k}$$

Now, since $f \in C([0,1])$, we can say that f is uniformly continuous on $[0,1]$ and so

$$\begin{aligned}
|f(x) - B_n(f;x)| &= \left| f(x) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \right| \\
&\leq \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \\
&= S_1 + S_2
\end{aligned}$$

We wish to construct S_1 and S_2 so that S_1 sums over all k such that

$$\left|x - \frac{k}{n}\right| > \delta$$

and so that S_2 sums over the rest of k between 0 and n. That is

$$\left|x - \frac{k}{n}\right| \leq \delta$$

Now, to bound S_1, we have

$$\left|f(x) - f\left(\frac{k}{n}\right)\right| \leq 2 \cdot \|f\|_\infty = 2 \sup_{x \in [0,1]} |f(x)|$$

and so

$$\begin{aligned} S_1 &\leq 2\|f\|_\infty \frac{x(1-x)}{n\delta^2} \\ &\leq \frac{\|f\|_\infty}{2n\delta} \end{aligned}$$

by the fact that $x(1-x)$ has maximum value $1/4$. Now, to bound S_2 we choose $\epsilon > 0$ and there exists $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$$

Now, in the second sum we know that

$$\left|x - \frac{k}{n}\right| < \delta \Longrightarrow \left|f(x) - f\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2}$$

so that

$$S_2 \leq \frac{\epsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \frac{\epsilon}{2}$$

Then

$$S_1 + S_2 < \frac{\epsilon}{2} + \frac{\|f\|_\infty}{2n\delta^2}$$

Finally, let n be large so that

$$\frac{\|f\|_\infty}{2n\delta^2} < \frac{\epsilon}{2}$$

which gives that

$$S_1 + S_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Section 6.8

Inverse & Implicit Function Theorem In $\mathbb{R}^n$

Theorem 95 (Inverse Function Theorem). *Let $\Omega \subseteq \mathbb{R}^n$ and let $F : \Omega \to \mathbb{R}^n$ be such that $F \in C^1(\Omega)$ where $F = [F_1, \ldots, F_n]$ and $F_i \in C^1(\Omega)$ for each $1 \leq i \leq n$. Let $a \in \Omega$ and let $DF(a)$ be invertible where*

$$(DF)_{ij} = \frac{\partial F_i}{\partial x_j}$$

and if we let $F(a) = b \in \mathbb{R}^n$, then

(i) There exists open sets $U, V \subseteq \Omega$ such that $a \in U, b \in V$ and $F : U \to V$ is a bijection.

(ii) The inverse function $G = F^{-1}$ defined by $G(F(x)) = x$ is contained in $C^1(V)$.

Proof.
Part (i).
Let $A = DF(a)$ and let

$$\epsilon = \frac{1}{4\|A^{-1}\|_{op}}$$

and $x \mapsto DF(x)$ is a continuous map from $\Omega \to Mat_{n\times n}(\mathbb{R})$. There exists an open ball U containing a such that

$$\|DF(x) - A\|_{op} < 2\epsilon$$

for any $x \in U$. So let $x, x+h \in U$ and let $f(t) = F(x+th) - t \cdot Ah$ for $t \in [0,1]$. Also, bear in mind that x and h are both vectors. Now we look at

$$\|f'(t)\| = \|DF(x+th) \cdot h - Ah\|| \leq 2\epsilon \cdot \|h\|$$

and now

$$2\epsilon\|h\| = 2\epsilon\|A^{-1} \cdot A \cdot h\| \leq 2\epsilon\|A^{-1}\|_{op} \cdot \|Ah\| = \frac{\|Ah\|}{2}$$

by our definition of ϵ. Now, we have

$$\|f'(t)\| \leq \frac{1}{2}\|Ah\|$$

and by the Generalized Intermediate Value Theorem we get

$$\|f(1) - f(0)\| \leq \frac{1}{2}\|Ah\|$$

and by our definition of f, this is the same as

$$\|F(x+h) - F(x) - Ah\| \leq \frac{1}{2}\|Ah\|$$

and so by the Triangle Inequality,

$$\|F(x+h) - F(x)\| \geq \|Ah\|\left(1 - \frac{1}{2}\right) = \frac{\|Ah\|}{2}$$

and since

$$\|Ah\| = \frac{1}{4\epsilon}$$

and since by the computation of $2\epsilon\|h\|$ which gives

$$\|h\| \leq \frac{\|Ah\|}{4\epsilon} \Longrightarrow \|Ah\| \geq 4\epsilon\|h\|$$

we get

$$\|F(x+h) - F(x)\| \geq 2\epsilon\|h\|$$

and this implies that F is 1-to-1 on U.
We now claim that if $x_0 \in U$ and $r > 0$ is such that $\overline{B}(x_0, r) \subseteq U$ then

$$B(F(x_0), \epsilon r) \subseteq F(B(x_0, r))$$

to this end, let $S = \overline{B}(x_0, r)$ and let

$$\|y - F(x_0)\| < \epsilon r$$

For $x \in S$, define

$$\psi(x) = \|y - F(x)\|^2$$

It suffices to show (for the claim) that $\psi(x) = 0$ for some $x \in S$. First, ψ is continuous since F is continuous and y is fixed and S is compact, so ψ takes a minimum and a maximum on S and in particular, ψ takes a minimum at say $x = x_1$. Second, ψ is differentiable particularly at x_1 and

$$\psi'(x) = DF(x) \cdot (y - F(x))$$

noticing that 0 denotes the zero vector $\vec{0}$ and at the minimum,

$$\psi'(x_1) = 0 = DF(x_1) \cdot (y - F(x_1))$$

and since $DF(x_1)$ is invertible, we must have that $y = F(x_1)$ and so $\psi(x_1) = 0$ as required.
We have just shown that every point in $F(U)$ is an interior point. Since $B(F(x), \epsilon r) \subseteq F(U)$, take $V = F(U)$ and we have yielded (i) of the theorem. Part (ii).

Define $G = F^{-1}$ by $G(F(x)) = x$. Then $G \in C^1(V)$. Let $y, y + k \in V$. Now, let $x = G(y) \in U$ and let $h = G(y + k) - G(y)$ so that $x + h = G(x + k)$. Then $x + h \in U$. Now, $DF(x)$ is invertible. Also,

$$\|DF(x) - A\|_{op} < 2\epsilon$$

on U, so let $B(x) = [DF(x)]^{-1}$. Now

$$(*) \qquad k = F(x + h) - F(x) = DF(x) \cdot h + r(h)$$

where

$$\frac{\|r(h)\|}{\|h\|} \to 0$$

as $\|h\| \to 0$. Apply $B = B(x)$ to both sides of $(*)$ so that

$$Bk = [B(x) \cdot DF(x)] \cdot h + B \cdot r(h) = h + B \cdot r(h)$$

since $B \cdot DF(x) = Id$. Then

$$h = G(y + k) - G(y) = B \cdot k - B \cdot r(h)$$

We've already proven that

$$\|F(x + h) - F(x)\| > \frac{1}{2}\|Ah\| \geq 2\epsilon\|h\|$$

which implied that the map is 1-to-1. This now becomes

$$\|k\| \geq 2\epsilon\|h\|$$

if $\|k\| \to 0$, then $\|h\| \to 0$ since ϵ is fixed. Therefore, the map G is continuous at y since we've shown that

$$\lim_{\|k\| \to 0} G(y + k) = G(y)$$

Now, to show that G is differentiable, we look at

$$\frac{\|B \cdot r(h)\|}{\|k\|} \leq \frac{\|B\|_{op} \cdot \|r(h)\|}{2\epsilon\|h\|} = \frac{\|B\|_{op}}{2\epsilon} \cdot \frac{\|r(h)\|}{\|h\|} \to 0$$

as $\|k\| \to 0$, since

$$\frac{\|r(h)\|}{\|h\|} \to 0$$

by definition of remainder. Thus, G is differentiable at y and its derivative is

$$DG(y) = [DF(G(y))]^{-1} \qquad \square$$

THEOREM 96 (Implicit Function Theorem). *Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and let $E \subseteq \mathbb{R}^{n+m}$ and $F : E \to \mathbb{R}^n$ be such that $F \in C^1(E)$. Now, let $(a, b) \in E$ be such that $F(a, b) = 0$ and let $M = DF(a, b)$ where M is an $n \times (n + m)$ matrix so write*

$$DF = (D_x F \mid D_y F)$$

where $D_x F$ is $n \times n$ and $D_y F$ is $n \times m$. Also,

$$(D_x F)_{ij} = \left(\frac{\partial F_i}{\partial x_j}\right) \qquad (D_y F)_{ij} = \left(\frac{\partial F_i}{\partial y_j}\right)$$

Now suppose that $D_x F$ is invertible, then there exists an open set $W \subseteq \mathbb{R}^n$ containing b and a ***unique*** *function $G : W \to \mathbb{R}^n$ such that $G(b) = a$ and*

$$F(G(y), y) \equiv 0$$

where $G(y) = x$ is then dependent and y is independent. Moreover, $G \in C^1(W)$.

COROLLARY 97. *We have*

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_m)} = [-D_x F]^{-1} \cdot [D_y F]$$

CHAPTER 7

LINEAR OPERATORS & THE OPERATOR NORM

SECTION 7.1

THE HAHN-BANACH THEOREM

THEOREM 98 (Hahn-Banach Theorem). *Let L be a linear subspace of a normed linear space X. Let $f(x)$ be a linear functional defined on L. Then $f(x)$ can be extended to a linear functional F on X such that*

$$\|F\|_{op,X} = \|f\|_{op,L}$$

Note that ***extended*** *means that $F|_L = f$. That is, for any $x \in L$, $F(x) = f(x)$.*
Proof.
We shall give a proof in case X is separable (contains a countable dense subset). Let $x_0 \in X, x_0 \notin L$ and let

$$L_1 = \{x_1 + tx_0 : t \in \mathbb{R}, x_1 \in L\}$$

Now, let $y \in L_1$ which implies that

$$y = tx_0 + x \qquad x \in L, t \in \mathbb{R}$$

So,

$$F(y) = tF(x_0) + f(x)$$

and let $F(x_0) = -C$ which gives

$$F(y) = f(x) - Ct$$

We want to show that

$$\|F\|_{op,X} \le \|f\|_{op,L}$$

We should then have that

$$|F(y)| = |f(x) - Ct| \le \|f\| \cdot \|x + tx_0\| = \|f\| \cdot |t| \cdot \left\|x_0 + \frac{x}{t}\right\|$$

assuming $t \neq 0$, *so let* $z = \frac{x}{t}$. *Then we rewrite*

$$|F(y)| = t \left| f\left(\frac{x}{t}\right) - c \right| = t \cdot |f(z) - c|$$

so that

$$|f(z) - c| \le \|f\| \cdot \|z + x_0\|$$

then

$$-f(z) - \|f\| \cdot \|z + x_0\| \le f(z) - C \le \|f\|_{op} \cdot \|z + x_0\|_X - f(z)$$

which implies that

$$(*) \quad f(z) - \|f\|_{op} \cdot \|z + x_0\|_X \le C \le f(z) + \|f\|_{op} \cdot \|z + x_0\|_X$$

Now, if $(*)$ *holds, then*

$$\|F\|_{op,X} \le \|f\|_{op,L}$$

and we can do an induction step so that $(*)$ *should hold for any* $z \in L$. *We now want to show that*

$$\sup_{z \in L}(f(z) - \|f\|_{op} \cdot \|z + x_0\|_X) \le \inf_{z \in L}(f(z) + \|f\|_{op} \cdot \|z + x_0\|_X)$$

so we claim that if $z_1, z_2 \in L$, *then*

$$f(z_2) + \|f\| \cdot \|z_2 + x_0\| \ge f(z_1) - \|f\| \cdot \|z_1 + x_0\|$$

To this end, we write

$$\begin{aligned} f(z_1) - f(z_2) \le f(z_1 - z_2) &\le \|f\|_{op} \cdot \|z_2 - z_2\| = \|f\| \cdot \|(z_1 + x_0) - (z_2 + x_0)\| \\ &\le \|f\|(\|(z_1 + x_0)\| + \|(z_2 + x_0)\|) \end{aligned}$$

and our claim follows. So, we can now extend f *to*

$$\{L + tx_0 : t \in \mathbb{R}\}$$

such that $\|F\|_{L_1} \le \|f\|_L$. *Let us take a countable dense subset* $x_1, \ldots, x_n \in X$ *where the* x_i*'s are linearly independent and not in L. First, extend* f *to* F_1 *on* $L + \langle x_1 \rangle = L_1$. *Then extend to* F_2 *on* $L_1 + \langle x_2 \rangle = L_2$, *and continue until we obtain* F_n *defined on* $L_{n-1} + \langle x_n \rangle = L_n$. *At each step, we have that*

$$\|F\|_{op,X} = \|f\|_{op,L}$$

This way, we obtain a bounded linear functional F, *defined on a dense subset of* X *such that* $\|F\| = \|f\|$. *It is now left as an excercise to show that on the rest of* X, *we can define* F *by continuity, so that* $x_n \to z$ *gives*

$$F(z) = \lim_{n \to \infty} F(x_n)$$

and

$$|F(x_n) - f(x_m)| \le \|f\| \cdot \|x_n - x_m\|$$

where $|F(x_n) - f(x_m)|, \|x_n - x_m\| \to 0$. *This will define a bounded linear functional* F *on* X.

Corollary 99. *Let $x_0 \in X$, where $x_0 \neq 0$ and let $M > 0$, then there exists $f \in X^*$ where f is a bounded linear functional on X such that*

$$\|f\|_{op} = M$$

and

$$f(x_0) = M \cdot \|x_0\| = \|f\|_{op} \cdot \|x_0\|_X$$

Proof.
Define f on $\langle x_0 \rangle$ by $F(tx_0) = M \cdot t \cdot \|x_0\|$ and the rest follows from Hahn Banach.

We recall that for any $x_0 \neq 0, x_0 \in X$, there exists a linear functional $A \in X^*$, such that

$$|Ax_0| = \|A\|_{op} \cdot \|x_0\|_X$$

and if X is a normed linear space, then so is X^*. Also, we can define a linear functional on X^* by $A \in X^*$,

$$f_{x_0}(A) = Ax_0$$

with

$$(\lambda_1 A_1 + \lambda_2 A_2)(x_0) = \lambda_1 A_1 x_0 + \lambda_2 A_2 x_0$$

This way, X can be realized as a linear submanifold of X^{**}. So on X^*, we have

$$\|L_{x_0}\|_{op} = \sup_{A \in X^*} \frac{|Ax_0|}{\|A\|_{op}} \leq \|x_0\|_X$$

and

$$|L_{x_0} A| = |Ax_0|$$

with $\|A\|_{op} \neq 0$. Now, Hahn-Banach implies what we just recalled. This shows that

$$\|L_{x_0}\|_{op} = \|x_0\|_X$$

So, X can be isometrically embedded in X^{**}.

Section 7.2
Examples

Also recall that if X is reflexive if $X^{**} = X$.
Example 30. *We now consider the norms of some linear operators.*

(i) Let $X = C([0,1])$ and take

$$(Tf)(x) = g(x)f(x)$$

and we ask for which g is T continuous. The answer to this is continuous g. To find $\|T\|_{op}$, we let $g \in C([0,1])$ and let $f \equiv 1$ so that $Tf = T1 = g \in C([0,1])$. Then

$$\|T\|_{op} = \|g\|_\infty = \sup_x |g(x)|$$

Now we claim that since

$$\|T\|_{op} = \sup \frac{|Tf|}{\|f\|}$$

and so

$$\sup_x |f(x)g(x)| \leq \sup_x |f(x)| \sup_x |g(x)| = \|f\|_\infty \cdot \sup_x |g(x)|$$

Now, find $f(x)$ such that $\sup |f \cdot g| = \sup |f| \sup |g|$. Just take $f \equiv 1$!

(ii) *Let $X = l_2$ and take*

$$Tx = (\alpha_1 x_1, \dots, \alpha_n x_n, \dots)$$

and we ask for which $\alpha = (\alpha_1, \dots, \alpha_n, \dots)$ is T a bounded linear operator. Now, T is bounded if and only if $\sup_k |\alpha_k| \le M < \infty$. This is true since if $\alpha_j \le M$ for each j, then

$$\sum_{k=1}^{\infty} |\alpha_k x_k|^2 \le M^2 \sum_{k=1}^{\infty} |x_k|^2$$

and so

$$\|Tx\|_{l_2}^2 \le M^2 \|x\|_{l_2}^2$$

Now, suppose that $\sup_k |\alpha_k| = \infty$, then take $k_1 < \cdots < k_n < \cdots$ such that $|\alpha_{k_n}| > n$ then let

$$x = \left(0, \dots, 0, 1, 0, \dots, 0, \frac{1}{2}, 0, \dots, 0, \frac{1}{n}, 0, \dots\right)$$

where terms only appear at the k_j index and zeros everywhere else. Then

$$\|x\|_2^2 = \sum_{k=1}^{\infty} \frac{1}{n^2} < \infty$$

now,

$$Tx = \left(0, \dots, 0, \alpha_{k_1}, 0, \dots, 0, \frac{\alpha_{k_2}}{2}, 0, \dots, 0, \frac{\alpha_{k_n}}{n}, 0, \dots\right) \notin l_2$$

so T is not well defined since

$$|\alpha_{k_n}| \ge 1$$

and it is left as an excercise to show that

$$\|T\|_{op} = \sup_k |\alpha_k|$$

and note that we have already proven "$\le$". Take a sequence of "test vectors" $x^{(j)} \in l_2$ such that

$$\frac{\|Tx^{(j)}\|_2}{\|x^{(j)}\|_2} \to \sup_k |a_k|$$

Suppose taht $\alpha_j \neq 0$ for all j, then

$$T^{-1}x = \left(\frac{x_1}{\alpha 1}, \dots, \frac{x_n}{\alpha_n}, \dots\right)$$

and whe ask when T, T^{-1} are both continuous. So

$$0 < m < |\alpha_j| \le M < \infty$$

for any j and so

$$\frac{1}{|\alpha_j|} < M_2 < \infty$$

(iii) *Consider the shift operator*

$$Tx = (0, x_1, \dots, x_n, \dots)$$

that shifts to the left so that the inverse is the right shift operator S. Then we define the left inverse to be S such that $T \circ S = Id$ and we define the right inverse to be S such that $S \circ T = Id$. Now, if we let

$$Sx = (x_2, \dots, x_n, \dots)$$

then, S is a left inverse, but T has no right inverse since we "lose" the x_1 term after applying S. Thus, $T \circ S \neq Id$ since T is not onto l_2.

(iv) Let $Y = C^1([0,1])$ and let

$$\|f\|_{C^1} = \sup_x |f(x)| + \sup_x |f'(x)|$$

and let $T : C^1 \to C^0$ where

$$(Tf)(x) = f'(x)$$

Then T is bounded for $\|T_{op}\| \leq 1$. Now, let

$$Sf(x) = \int_0^x f(s)ds$$

where $S : C^0 \to C^1$ then $T(S(f)) = f$ so that S is the right inverse of T. However, it is left as an excercise to show that T has no left inverse. But, let $Y \subseteq C^1$ defined by

$$Y = \{f \in C^1 : f(0) = 0\}$$

then, S is a left inverse on Y.

Fin.

Good Luck On The Final!

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