McGill University
Math 354: Honors Calculus 3

Various things about metric spaces

I. Application of Stone-Weierstrass theorem.

Let $\mathbb{T} = \mathbf{R}(\text{mod} 2\pi)$ be the unit circle (or 1-dimensional torus). Consider the set $C(\mathbb{T})$ of continuous functions on $\mathbb{T}$ (they may be identified with continuous 2π-periodic functions on $\mathbf{R}$). Among such functions you have $\sin(nx)$ and $\cos(nx), n \in \mathbf{N}$; when $n = 0$ we get constants.

Theorem 1. The set $\mathcal{P}$ consisting of linear combinations of the form

$$c + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

is dense in $C(\mathbb{T})$ (where $a_k, b_k, c \in \mathbf{R}$).

Theorem 1 is an important result in the theory of *Fourier series* which you will study in Math 355. Theorem 1 is proved by an application of Stone-Weierstrass approximation theorem. Since $\mathbb{T}$ is compact, it suffices to show the following

Lemma 2. $\mathcal{P}$ is a unital separating subalgebra of $C(\mathbb{T})$.

Proof of Lemma 2. Clearly, $\mathcal{P}$ is unital (constants belong to $\mathcal{P}$). We leave the proof of the separating property as an exercise (to brush up your trigonometric skills!) Also, all functions in $\mathcal{P}$ are continuous, so $\mathcal{P} \subset C(\mathbb{T})$. It remains to show that $\mathcal{P}$ is an algebra. This would follow if we show that products $\sin(mx)\sin(nx), \sin(mx)\cos(nx), \cos(mx)\sin(nx), \cos(mx)\cos(nx)$ all belong to $\mathcal{P}$ for arbitrary $m, n \in \mathbf{N}$. The statement follows immediately from the so-called product formulas for sin and cos:

$$\begin{aligned}
\cos(a)\cos(b) &= \frac{1}{2}(\cos(a+b) + \cos(a-b));\\
\sin(a)\sin(b) &= \frac{1}{2}(\cos(a-b) - \cos(a+b));\\
\sin(a)\cos(b) &= \frac{1}{2}(\sin(a+b) + \sin(a-b));\\
\cos(a)\sin(b) &= \frac{1}{2}(\sin(a+b) - \sin(a-b)).
\end{aligned}$$

Remark. All these formulas can be easily derived by considering $\mathbf{C}$-valued functions $e^{int} := \cos(nt) + i\sin(nt)$, where $i = \sqrt{-1}$, but we gave a "real-variable" proof instead.

II. Second countable spaces.

Proposition 3. In a separable metric space, any open set can be written as a countable union of open balls. Also, every separable metric space X is *second countable*, i.e. in such a space there exists a countable family of open subsets U_n (called a *countable base*), such that every open set V in X satisfies $V = \cup_{U_n \subset V} U_n$.

Proof. (S. Drury, Theorem 27, p. 38). Recall that a space is called *separable* if it has a countable dense subset $\{x_n\}_{n=1}^{\infty}$. It suffices to show that every open set in X can be written as a countable union of open balls centered at x_n-s with rational radii (the set of such balls is countable).

Let V be an open subset of X. For every $x_n \in V$, let $t_n = \text{dist}(x_n, X \setminus V)$; t_n is positive since $X \setminus V$ is closed and $x_n \notin X \setminus V$. Let q_n be a rational number in the interval $[t_n/2, 2t_n/3]$. It suffices to show that

$$V \subseteq \cup_n U(x_n, q_n), \tag{1}$$

since obviously $U(x_n, q_n) \subset U(x_n, 2t_n/3) \subset V$.

To prove (1), we remark that $\cup_n U(x_n, q_n)$ contains the set $\cup_n U(x_n, t_n/2)$. Accordingly, it suffices to show that

$$V \subseteq \cup_n U(x_n, t_n/2). \tag{2}$$

Let $y \in V$, and let $t := \mathrm{dist}(y, X \setminus V) > 0$. Since x_n-s are dense, there exists n such that $d(y, x_n) < t/3$. By the solution of Problem 5 in Assignment 1, we have

$$|\mathrm{dist}(y, X \setminus V) - \mathrm{dist}(x_n, X \setminus V)| \le d(y, x_n) < t/3,$$

hence $t_n = \mathrm{dist}(x_n, X \setminus V) > 2t/3$. Therefore,

$$d(y, x_n) < \frac{t}{3} = \frac{1}{2}\left(\frac{2t}{3}\right) < \frac{t_n}{2},$$

and thus $y \in U(x_n, t_n/2)$. This finishes the proof of (2).

III. Countable compactness.

The following material can be found in S. Drury's Math 354 notes (Theorem 74, p. 100).

Definition 4. A metric space X is called *countably compact* if every *countable* open cover contains a finite subcover.

Note that in the usual definition, arbitrary covers are allowed. The new definition may in principle be easier to check than the old one. We want to show that for separable metric spaces, the two definitions are equivalent:

Proposition 5. If X is separable and countably compact, then it is compact.

The proof is an application of Proposition 3. Indeed, let U_n be a countable set of open balls (centered at x_n-s with rational radii) s.t. every open set in X is a union of a subset of U_n-s. Then $V_\alpha = \cup_{U_n \subset V_\alpha} U_n$, and it is easy to see that $X = \cup_k U_{n_k}$. Note that each of U_{n_k}-s is contained in some V_α. By countable compactness, we can chose a finite subcover, so

$$X \subseteq \cup_{k=1}^m U_{n_k} \subseteq \cup_{k=1}^m V_{\alpha_k},$$

where $U_{m_k} \subset V_{\alpha_k}$. We have thus found a finite subcover, finishing the proof.

IV. Normal and Hausdorff spaces.

A *topological space* X is a set, together with collections of open and closed sets that satisfy the usual properties (A is open iff $X \setminus A$ is closed; countable union of open sets is open; finite intersection of open sets is open; $\emptyset$ and X are both open and closed, etc). Metric spaces are examples of topological spaces, but there exists topological spaces that are not metric spaces.

Definition 6. A topological space X is called *Hausdorff* iff for every two points $x \neq y \in X$, there exists an open set $U, x \in U$, and another open set $V, y \in V$ such that $U \cap V = \emptyset$.

Definition 7. A topological space X is called *normal* iff for any two closed sets A, B such that $A \cap B = \emptyset$, there exists an open set $U, A \subset U$, and another open set $V, B \subset V$ such that $U \cap V = \emptyset$.

Clearly, a normal topological space is Hausdorff if one-point sets $\{x\}$ are closed. We shall prove the following

Proposition 8. Metric spaces are Hausdorff and normal.

Proof. Let $x \neq y \in X$, and let $r = d(x, y) > 0$. To verify the property of Definition 6, we can take $U = U(x, r/3)$ and $V = U(y, r/3)$.

Next, let A, B be disjoint closed subsets of X. Consider the continuous function $F : X \to \mathbf{R}$ defined in exercise 8, Assignment 2. Then $A = F^{-1}(\{0\})$ and $B = F^{-1}(\{1\})$. We can take

$$U = F^{-1}((-\infty, 1/3)), \quad V = F^{-1}((2/3, +\infty)).$$

Then U and V are open since F is continuous; $A \subset U$, $B \subset V$; and U and V are clearly disjoint. This shows that the property of the Definition 7 holds, QED.

The following theorem clarifies relationship between topological spaces and metric spaces:

Theorem 9 (Urysohn's metrization theorem). Let X be topological space that is second countable (has a countable base, cf. Proposition 3), and normal (cf. Definition 7). Then X is *metrizable*, i.e. one can define a distance d on X such that open and closed sets in X will coincide with the open and closed sets for the distance d.

Proofs of this theorem can be found in textbooks of point-set topology.