McGill University
Math 354: Honors Analysis 3

An estimate for the Intermediate Value theorem

We shall prove an estimate (Drury's Math 354 notes, theorem 137) that is used to prove the Intermediate Value theorem in normed vector spaces.

Theorem. Let X be a normed vector space, I an interval containing $J := [0, 1]$ in its interior, and $f : I \to X$ a differentiable function. Then

$$||f(1) - f(0)||_X \leq \sup_{t \in [0,1]} ||f'(t)||_X.$$

Proof. (Drury, p. 154). Let $C := \sup_{t\in[0,1]} ||f'(t)||_X$. We may assume without loss of generality that $C < \infty$, otherwise the estimate is trivially true.

Fix $\epsilon > 0$ and consider the set

$$A = \{t \in [0, 1] : ||f(t) - f(0)|| \leq (C + \epsilon)t\}.$$

It is easy to see that A is closed and that $0 \in A$. Next define

$$B = \{s \in [0, 1] : [0, s] \subset A\}.$$

Clearly, $0 \in B$. We claim that B is closed. It suffices to show that $J \setminus B$ is open. Indeed, if $s \notin B$, then there exists $t \in [0, s]$ such that $||f(t) - f(0)|| > (C + \epsilon)t$. If $t \in (0, s)$, let $\delta = s - t$. Then by definition of B we see that $(s - \delta/2, s + \delta/2) \subset J \setminus B$ (recall that $J = [0, 1]$). If $t = s$, we have $||f(s) - f(0)|| > (C + \epsilon)s$. Since we have continuous functions of s on both sides of the inequality, we find that the inequality continues to hold for all $t \in (s - \delta, s + \delta)$ for some $\delta > 0$, again implying that s is an interior point of $J \setminus B$, QED.

Lemma. B is open.

Lemma clearly implies the Theorem, since it follows that B is both open and closed, and nonempty. Since J is connected, we must have $B = J$, and so $||f(1) - f(0)|| \leq C + \epsilon$. Since ϵ was arbitrary, we are done.

Proof of the Lemma. Let $s \in B$, hence $s \in A$. We have

$$||f(s) - f(0)|| \leq (C + \epsilon)s. \tag{1}$$

We now use differentiability at s:

$$f(s + r) = f(s) + rf'(s) + \phi(r),$$

where $||\phi(r)/r||_X \to 0$ as $r \to 0$. Therefore, $\exists\delta > 0$ such that for $0 \leq r \leq \delta$ we have

$$||f(s + r) - f(s)|| \leq (C + \epsilon)r.$$

Combining with the (1) we find that

$$||f(s + r) - f(0)|| < (C + \epsilon)(s + r),$$

and so $[s, s + \delta] \subset B$ and B is open, QED.