McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 4 — due Friday, October 19

**Problem 1 (extra credit).** Let $X = C^1[0,1]$ denote the space of continuously differentiable functions on $[0,1]$.

a) Prove that the expression

$$||f||_2 = \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |f'(x)|.$$

defines a norm on $X$

b) Prove that $(X, ||\cdot||_2)$ is a complete metric space. Is it separable (does it contain a countable dense set)?

c) Prove that $||f||_2$ does not define the same topology on $X$ as the $d_\infty$ norm $\max_{x\in[0,1]} |f(x)|$.

**Solution:**

a) Clearly $\|\cdot\|_2$ satisfies $\|f\|_2 \geq 0$ and $\|\lambda f\|_2 = |\lambda|\|f\|_2$. If $f \equiv 0$ then $f' \equiv 0$ and $\|f\|_2 = 0$. Now, if $\|f\|_2 = 0$, both $\max_{x\in[0,1]} |f(x)| = 0$ and $\max_{x\in[0,1]} |f'(x)| = 0$. This implies $f \equiv 0$, $f' \equiv 0$ so $f$ is the zero function. The triangle inequality holds since

$$\begin{aligned}\|f+g\|_2 &= \max_{x\in[0,1]} |f+g(x)| + \max_{x\in[0,1]} |(f+g)'(x)| \leq \max_{x\in[0,1]} (|f(x)| + |g(x)|) + \max_{x\in[0,1]} (|f'(x)| + |g'(x)|) \\ &\leq \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |g(x)| + \max_{x\in[0,1]} |f'(x)| + \max_{x\in[0,1]} |g'(x)| = \|f\|_2 + \|g\|_2.\end{aligned}$$

b)Let $(f_n)$ be a Cauchy sequence in $\|\cdot\|_2$. Let $\epsilon > 0$. There exists $n \in N$ such that for all $n, m \geq N$

$$|f_n(x) - f_m(x)| + |f'_n(x) - f'_m(x)| < \epsilon$$

for all $x \in [0,1]$. In particular, $d_\infty(f'_n, f'_m) < \epsilon$, so $(f'_n)$ is Cauchy in $d_\infty$, and therefore there exists a uniform limit, say $g$ (continuous). Let

$$f(x) = \int_0^x g(t)dt.$$

We have that, for $n \geq N' \geq N$ and for all $x \in [0,1]$,

$$-\frac{\epsilon}{2} < f'_n(x) - g(x) < \frac{\epsilon}{2}.$$

Integrating over $[0,1]$ we get $-\frac{\epsilon}{2} < f_n(x) - f(x) < \frac{\epsilon}{2}$ and thus

$$\|f_n - f\|_2 < \epsilon.$$

By Stone-Weierstrass theorem we know that polynomials with rational coefficients form a countable dense subset of $C([0,1])$. Let $f \in C^1([0,1])$ and $\epsilon > 0$. We have that $f' \in C([0,1])$ so there exists a polynomial with rational coefficients, $q(x)$, such that

$$d_\infty(f', q) < \frac{\epsilon}{2}.$$

Then, for all $x \in [0,1]$, $-\frac{\epsilon}{2} < f'(x) - q(x) < \frac{\epsilon}{2}$ and therefore $-\frac{\epsilon}{2} < f(x) - Q(x) < \frac{\epsilon}{2}$, where $Q(x)$ is a polynomial with rational coefficients such that $Q'(x) = q(x)$. Then, $\|f - Q\|_2 < \epsilon$.

c)We will show that any ball about $f \equiv 0$ in $d_\infty$ contains functions with arbitrarily big derivative and, therefore, such ball cannot be contained in any ball about 0 in $\|\cdot\|_2$.
Let $\epsilon > 0$ and consider $B^\infty(0, \epsilon)$ the ball about $f \equiv 0$ in $d_\infty$. Let

$$f_n(x) = \frac{1}{n}\sin(n^2 x),$$

$x \in [0,1]$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then for all $n \geq N$, $\|f_n\|_\infty < \epsilon$, so $f_n \in B^\infty(0,\epsilon)$. However,

$$f_n'(x) = n\cos(n^2 x),$$

so $\|f_n\|_2 = n + 1$.

**Problem 2.** Let $f_n : [0,1] \to \mathbf{R}$ be a sequence of continuously differentiable functions satisfying

$$|f_n(x)| \leq M, |f_n'(x)| \leq M, \qquad \forall x \in [0,1],\ \forall n \in \mathbf{N}.$$

Prove that $\{f_n\}$ has a uniformly convergent subsequence.
**Solution.** By Arzela-Ascoli theorem, it suffices to show that $\{f_n\}$ is uniformly bounded (true by assumption), and (uniformly) equicontinuous. Accordingly, given $\epsilon > 0$, let $\delta = \epsilon/M$. Then for any $n$, and for any $x < y \in [0,1]$ such that $|y - x| < \delta$, we have by the intermediate value theorem

$$|f_n(y) - f_n(x)| \leq \delta \cdot \sup_{z \in [x,y]} |f_n'(z)| < M \cdot \frac{\epsilon}{M} = \epsilon,$$

proving uniform equicontinuity. QED
**Problem 3.** Determine whether the following sets of functions are sequentially compact in $C[0,1]$:

a) $\{(ax)^n\}, n \in \mathbf{N}, a > 0$.

b) $\{\sin(x+n)\}, n \in \mathbf{N}$.

c) $\{e^{x-a}\}, a > 0$.

d) $\{f \in C^2[0,1] : |f(x)| < B_0, |f'(x)| < B_1, |f''(x)| < B_2\}$.

e) (extra credit) $\{f \in C^2[0,1] : |f(x)| < B_0, |f''(x)| < B_2\}$.

f) $\{f \in C^2[0,1] : |f'(x)| < B_1, |f''(x)| < B_2\}$.

**Solution:**

a) $\{(ax)^n\}, n \in \mathbf{N}, a > 0$. Clearly, for $a > 1$, the sequence $f_n(1) = a^n$ is not bounded, so the answer is NO. Also, for $a < 1$, $f_n(x) \to 0$ uniformly on $[0,1]$, so the answer is YES. If $a = 1$, then $f_n(x) = x^n$ was considered in class. The answer is NO, since the limit function is discontinuous at $x = 1$.

b) $\{\sin(x+n)\}, n \in \mathbf{N}$. The sequence of functions is uniformly bounded, and has uniformly bounded derivatives $f'_n(x) = \cos(x+n)$. The answer is YES by Problem 2.

c) $\{e^{x-a}\}, a > 0$. The sequence of functions is uniformly bounded, and has uniformly bounded derivatives $f'_a(x) = e^{x-a}$. The answer is YES by Problem 2.

d) $\{f \in C^2[0,1] : |f(x)| < B_0, |f'(x)| < B_1, |f''(x)| < B_2\}$. YES by Problem 2.

e) (extra credit) $\{f \in C^2[0,1] : |f(x)| < B_0, |f''(x)| < B_2\}$. The answer is YES. The two conditions imply a uniform bound on the first derivative, then we can use Problem d). The proof will be provided separately.

f) $\{f \in C^2[0,1] : |f'(x)| < B_1, |f''(x)| < B_2\}$. NO, since the sequence is not necessarily uniformly bounded (e.g. arbitrary constant satisfies both conditions, and $f_n(x) = n$ has no convergent subsequence).