McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 2 — Solutions to selected problems

Problem 1. Let K be the Cantor set. Prove that

a) Prove that the endpoints of the intervals that appear in the construction of K, i.e. the points $0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \ldots$ are everywhere dense in K;

b) **(extra credit).** The numbers of the form $\{t_1 + t_2 : t_1, t_2 \in K\}$ fill the whole interval $[0, 2]$.

Solution: (i) After n iterations of removing "middle-third" intervals, all the remaining intervals have length $1/3^n$. Given $\epsilon > 0$, choose n such that $1/3^n < \epsilon/2$. Let $x \in K$. Then x lies in one of the n-th generation intervals, which by the choice of n is completely contained in the ϵ-neighborhood of x, together with its endpoints, proving the density.

(ii) We first assume the following key

Lemma. Let $[a_{2k-1}, a_{2k}], 1 \leq k \leq 2^n$ be the intervals remaining after n-th iteration. Then every rational number in $[0, 2]$ of the form $b/(3^n), 0 \leq 2 \cdot 3^n, b \in \mathbf{Z}$ can be represented as $a_i + a_j$ for some i and j.

We first show that Lemma implies the result of (ii). Indeed, let $x \in [0, 2]$, and let $b_n/3^n$ be a sequence of ternary approximations to x. Then by Lemma $b_n/3^n = x_n + y_n$, where x_n, y_n are endpoints of the intervals. The sequence x_n in $[0, 1]$ has a convergent subsequence (by compactness) that we denote $\{x_k\}, x_k \to p$ as $k \to \infty$. Since K is closed, $p \in K$. Since $x_n + y_n \to x$, we also have $y_n \to q = x - p$, and $x - p \in K$ as well since K is closed. Thus, $x = p + q$, with $p, q \in K$, QED.

Proof of the Lemma: It is easy to show by induction that the enpodints of the n-th generation intervals, different from 0 and 1, have the form

$$\frac{\alpha_n}{3^n} + \frac{2\alpha_{n-1}}{3^{n-1}} + \frac{2\alpha_{n-2}}{3^{n-2}} + \ldots + \frac{2\alpha_1}{3}, \tag{1}$$

where $\alpha_n \in \{0, 1, 2, 3\}$, and $\alpha_j \in \{0, 1\}$ for $j < n$. You can check it easily for $n = 1, 2$, then use the fact that K is self-similar, i.e. any of the n-th generation intervals of K is isometric to K after multiplication by 3^n and translation to the origin.

Next, consider any rational number in $[0, 2]$ of the form $b/3^n, b \in \mathbf{Z}$. We need to show that b is a sum of two endpoints of the n-th generation intervals. We have $0 \leq b < 2 \cdot 3^n$ (since $b = 2 \cdot 3^n$ corresponds to $2 = 1 + 1$). Divide b with remainder by $2 \cdot 3^{n-1}$ to get $b = \beta_1 \cdot 2 \cdot 3^{n-1} + r_1$, where $\beta_1 =\in \{0, 1, 2\}$ and $0 \leq r_1 < 2 \cdot 3^{n-1}$. Next, divide r_1 with remainder by $2 \cdot 3^{n-2}$ to get $r_1 = \beta_2 \cdot 2 \cdot 3^{n-2} + r_2$, where $\beta_2 \in \{0, 1, 2\}$ and $0 \leq r_2 < 2 \cdot 3^{n-2}$. Continue the process until you get $r_{n-2} = \beta_{n-1} 2 \cdot 3 + r_{n-1}$, $\beta_{n-1} \in \{0, 1, 2\}$ and $r_{n-1} \in \{0, 1, 2, 3, 4, 5\}$.

Accordingly,

$$\frac{b}{3^n} = \frac{r_{n-1}}{3^n} + \frac{2\beta_{n-1}}{3^{n-1}} + \frac{2\beta_{n-2}}{3^{n-2}} + \ldots + \frac{2\beta_2}{3^2} + \frac{2\beta_1}{3}, \tag{2}$$

where β_j and r_{n-1} were defined above. We need to show that the expression (2) can be represented as a sum of two expressions of the form (1); we denote by α_j^1, α_j^2 the corresponding numerators in (1), $1 \leq j \leq n$.

We shall consider different possibilities for r_{n-1}.

$r_{n-1} = 0$: Choose $\alpha_n^1 = \alpha_n^2 = 0$. For $j < n$, if $\beta_j = 2$, choose $\alpha_j^1 = \alpha_j^2 = 1$; if $\beta_j = 1$, choose $\alpha_j^1 = 1, \alpha_j^2 = 0$; if $\beta_j = 0$, choose $\alpha_j^1 = \alpha_j^2 = 0$.

$r_{n-1} = 1$: Choose $\alpha_n^1 = 1, \alpha_n^2 = 0$. For $j < n$, choose as before.

$r_{n-1} = 2$: Choose $\alpha_n^1 = 1, \alpha_n^2 = 1$. For $j < n$, choose as before.
$r_{n-1} = 3$: Choose $\alpha_n^1 = 2, \alpha_n^2 = 1$. For $j < n$, choose as before.
$r_{n-1} = 4$: Choose $\alpha_n^1 = \alpha_n^2 = 2$. For $j < n$, choose as before.
$r_{n-1} = 5$: Choose $\alpha_n^1 = 3, \alpha_n^2 = 2$. For $j < n$, choose as before.
Since all the cases were considered, the proof is finished.
QED
Problem 2. Let $A_1, A_2, \ldots$ be subsets of X. We define $\limsup A_n$ by

$$\limsup A_n = \cap_{N=1}^{\infty} \left(\cup_{n \geq N} A_n \right),$$

and $\liminf A_n$ by

$$\liminf A_n = \cup_{N=1}^{\infty} \left(\cap_{n \geq N} A_n \right),$$

Prove that

a) $x \in \limsup A_n$ iff (if and only if) $x \in A_n$ for infinitely many n.

b) $x \in \liminf A_n$ iff there exists an N such that $x \in A_n, \forall n \geq N$ ($x \in A_n$ with finitely many exceptions).

Solution:
a) Suppose $x \in A_n$ for only finitely many indices n, say n_1, n_2, ..., n_k, $k \geq 1$. Let $M = n_k + 1$. Then $x \notin \cup_{n \geq M} A_n \supset (\cap_{N=1}^{\infty} \cup_{n \geq N} A_n)$, so $x \notin \limsup A_n$

Conversely, if $x \notin \limsup A_n$ then there exists M such that $x \notin \cup_{n \geq M} A_n$, hence $x \notin A_n$ $\forall n \geq M$. Thus, x belongs to, at most, A_1, A_2, ..., A_{M-1} , i.e., x belongs to finitely many sets A_n (maybe zero.)

b) If $x \in \liminf A_n$ then there exists $N \in \mathbb{N}$ such that $x \in \cap_{n \geq N}$; this means that $x \in A_n$ for all $n \geq N$. Thus, there exists $N \in \mathbb{N}$ such that $x \in A_n$, $\forall n \geq N$.

If there exists $N \in \mathbb{N}$ such that $x \in A_n$, $\forall n \geq N$, then $x \in \cap_{n \geq N}$. Since $\cap_{n \geq N} \subset \cup_{N=1}^{\infty} \cap_{n \geq N}$, then $x \in \liminf A_n$.

Problem 3. Let X be a topological space, and let $A \subset X$. Prove that

a) The interior A^{int} of A is the largest open set contained in A;

b) The closure $\overline{A}$ of A is the smallest closed set containing A.

Solution:
(a)$A^{int} = \{x \in A : \exists U_x \text{ open, such that } x \in U_x \subset A\}$. The set A^{int} is open since it can be written as

$$A^{int} = \bigcup_{x \in A} U_x.$$

Suppose there exists an open set B such that $A^{int} \subset B \subset A$. Let $x \in B$. Since B open, there exists an open set V_x such that $x \in V_x \subset B$. Thus we have $x \in V_x \subset A$, so $x \in A^{int}$. Then $B \subset A^{int}$, so $B = A^{int}$.
(b) It suffices to show that any closed set A containing M must also contain all contact points of M. There are two kinds of contact points of M: those that belong to M (those are obviously in A); and limit points of A. Now, for any limit point y of A, there is a sequence $\{x_n\}$ of points in M converging to y. Now, all x_n belong to A, and hence their limit y must also be in A, since A is

closed, finishing the proof.

Problem 4. Prove that a subspace A of a complete metric space X is complete if and only if A is closed.
Solution: Suppose A is closed. Let $\{x_n\}$ be a Cauchy sequence in A. Then $x_n \to y \in X$ as $n \to \infty$, by completeness of X. Also, y is a limit point of A, and so $y \in A$, since A is closed. It follows that A is complete.

If A is not closed, there exists a limit point y of A that is not in A. Let x_n be a point of A in $B(y, 1/n)$. It is easy to see that $\{x_n\}$ is a Cauchy sequence (both in A and in X), but it converges to $y \notin A$, so A is incomplete.

Problem 5. Let (X, d) be a metric space and Y an open subset of X. Show that if A is dense in X, then $A \cap Y$ is dense in Y. Give a counterexample to the previous statement when Y is closed.
Solution: Let $y \in Y$; choose arbitrary $\epsilon > 0$. Since Y is open, there exists $r > 0$, which we can choose to be *less than* ϵ, such that $B(y, r) \subset Y$. Since A is dense in X, there exists $a \in B(y, r) \cap A \subset Y \cap A$. It follows that there is a point from $Y \cap A$ in $B(y, r) \cap Y$, so $Y \cap A$ is dense in Y.

For a counterexample, let A be the set of points in $\mathbf{R}^2$ with *both* coordinates in $\mathbf{Q}$. Clearly, A is dense in X. Now, let Y be the straight line $\{(t, \sqrt{2}), t \in \mathbf{R}\}$. Then $A \cap Y = \emptyset$ since $\sqrt{2} \notin \mathbf{Q}$.

Problem 6, Jensen's inequality (extra credit). Let $f(x)$ be Riemann integrable on $[a, b]$; $f(x) \in [c, d], \forall x \in [a, b]$; and that $\phi(y)$ is convex on $[c, d]$. Prove that

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \leq \frac{1}{b-a}\int_a^b \phi(f(x))dx.$$

For simplicity, you can assume that f is continuous.
Solution. There are a few ways of doing this, like taking a measure theoretical or probabilistic approach, but we need to wait for MATH 355 for that. Instead we'll use what we know from Problem 10 (iv) for finite sequences and generalize it to the continuous case. In that problem we proved that

$$\phi\left(\sum_{k=1}^n t_k x_k\right) \leq \sum_{k=1}^n t_k \phi(x_k).$$

Since f is Riemann integrable we can choose a partition of $[a, b]$, $a = x_0 \leq x_1 \leq \ldots \leq x_n = b$ and then

$$\frac{1}{b-a}\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k)\frac{\Delta x_k}{b-a}.$$

Now since $\sum_{k=1}^n \frac{\Delta x_k}{b-a} = 1$ by definition we apply the finite inequality above to $\sum_{k=1}^n f(x_k)\frac{\Delta x_k}{b-a}$ and get

$$\phi\left(\sum_{k=1}^n f(x_k)\frac{\Delta x_k}{b-a}\right) \leq \sum_{k=1}^n \phi(f(x_k))\frac{\Delta x_k}{b-a}. \tag{3}$$

Taking n to infinity on the right gives

$$\frac{1}{b-a}\int_a^b \phi(f(x))dx.$$

Since the left-hand side of (3) is bounded above, we take n to infinity to get the desired result.
Problem 7. Let $a < b, b - a = 1$, and let $f \in C([a, b])$.

a) Prove that

$$p \to ||f||_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

is non-decreasing for $p \geq 1$. Hint: use the Hölder's inequality for functions.

b) In a), find $\lim_{p\to\infty} ||f||_p$.

c) Does the statement in a) hold if $b - a \neq 1$? Justify your answer.

Solution. (a) Let $\alpha \geq 0$ and $\beta > 1$ be arbitrary. Notice that

$$\|f^\alpha\|_1 = \int_a^b |f(x)|^\alpha dx = \|f\|_\alpha^\alpha.$$

Recall that $b - a = 1$, and let $\gamma \in [1, \infty]$. Then

$$\|1\|_\gamma = \left(\int_a^b dx \right)^{1/\gamma} = (b-a)^{1/\gamma} = 1.$$

Let $g(x) = 1$ and apply Hölder's inequality with $p = \beta$ and $q = \frac{\beta}{\beta - 1}$, the conjugate of p,

$$\begin{aligned} \|f\|_\alpha^\alpha = \|f^\alpha \cdot 1\|_1 \leq \|f^\alpha\|_\beta \|1\|_{\frac{\beta}{\beta-1}} &= \|f^\alpha\|_\beta \\ = \left(\int_a^b |f(x)|^{\alpha\beta} dx \right)^{1/\beta} &= \|f\|_{\alpha\beta}^\alpha. \end{aligned}$$

Taking the α root of both sides we get

$$\|f\|_\alpha \leq \|f\|_{\alpha\beta}$$

for all $\alpha \geq 1, \beta > 1$. So if $p' > p$, let $\beta = p'/p$, and $\alpha = p$, the last inequality gives that

$$\|f\|_p \leq \|f\|_{p \cdot \frac{p'}{p}} = \|f\|_{p'}$$

which is what we wanted, since with $b - a = 1$ we see that $\psi(t) = \|f\|_t$.

(b) We claim this limit is $\|f\|_\infty$. Recall that $b - a = 1$. So it is enough to show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ after this normalization. For this see the solution to Problem 8 (ii).

Problem 8 (extra credit). Let $f \in C([a, b]), f \not\equiv 0$. For $p \geq 1$, define

$$\phi(p) := ||f||_p^p = \int_a^b |f(x)|^p dx.$$

i) If $1 \leq r < p < s < \infty$, prove that

$$||f||_p \leq \max(||f||_r, ||f||_s).$$

ii) Prove that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

iii) Prove that $\log \phi(p)$ is convex.

Solution i), iii):

If $r < p < s$, choose $0 < \lambda < 1$ such that $p = \lambda r + (1-\lambda)s$. Then $\frac{1}{\lambda}$ and $\frac{1}{1-\lambda}$ are Hölder conjugates, and we can thus apply Hölder's inequality to obtain

$$\begin{aligned}
\|f\|_p^p &= \int_a^b |f|^p \\
&= \int_a^b |f|^{\lambda r}|f|^{(1-\lambda)s} \\
&\leq \left(\int_a^b |f|^{\lambda r \frac{1}{\lambda}}\right)^{\lambda} \left(\int_a^b |f|^{(1-\lambda)s\frac{1}{1-\lambda}}\right)^{1-\lambda} \\
&= \|f\|_r^{\lambda r}\|f\|_s^{(1-\lambda)s}.
\end{aligned}$$

Taking logarithms, we find that

$$\log \phi(p) \leq \lambda \log \phi(r) + (1-\lambda)\log \phi(s),$$

proving iii). Convexity easily implies i), since the point $(p, \phi(p))$ lies below the straight line connecting the points $(r, \phi(r))$ and $(s, \phi(s))$.

ii): Answer: $\lim_{p\to\infty} ||f||_p = ||f||_\infty := \sup_{x\in[a,b]} |f(x)|$. Let $||f||_\infty := C$.

We first remark that

$$\left(\int_a^b |f(x)|^p\right)^{1/p} \leq \left(\int_a^b C^p\right)^{1/p} = C \cdot |b-a|^{1/p} \to C$$

as $p \to \infty$. Sometimes it is also convenient to normalize the integrals as follows:

$$||f||_p := \left(\frac{1}{b-a}\int_a^b |f|^p\right)^{1/p}.$$

In the other direction, it suffices to show that for any $\epsilon > 0$,

$$\lim_{p\to\infty} ||f||_p > C - \epsilon. \tag{4}$$

To that end, let $|f(x_0)| = C$ for some $x_0 \in [a,b]$. Choose $\delta > 0$ such that $\max\{(x_0 - a), (b - x_0)\} > \delta$, and also $|f(y)| > C - \epsilon$, for all $y \in [x_0 - \delta, x_0 + \delta] \cap [a,b]$.

Then $||f||_p \geq (2\delta(C-\epsilon)^p)^{1/p} = (C-\epsilon)(2\delta)^{1/p}$. Since δ is fixed, taking the limit as $p \to \infty$ implies (4).

Problem 9 (extra credit). Give an example of a complete metric space X and a sequence of *nested* closed balls (i.e. $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots \supseteq B_n \supseteq \ldots$) such that

$$\cap_{n=1}^\infty B_n = \emptyset.$$

Hint: the radius of the balls need not go to zero.

Solution: Let X be the set $\{1, 2, 3, \ldots, n, \ldots\}$, and let the distance be defined by

$$d(m, n) = \begin{cases} 1 + 1/(m+n), & m \neq n; \\ 0, & m = n. \end{cases}$$

It is easy to see that d is a distance, since $1 < d(m, n) < 2$ for $m \neq n$. It is easy to see that d defines *discrete* topology on X (i.e. every point is an open set, and hence so is *any* subset of X). The only Cauchy sequences in such a topology are *stationary* sequences, i.e. $x_n = x$ for n large enough; it follows that X is complete.

Now, consider the closed ball $B(m, 1 + 1/(2m))$. It is easy to see that $d(m, n) \leq 1 + 1/(2m)$ iff $n \geq m$, so

$$B(m, 1 + 1/(2m)) = \{m, m+1, m+2, \ldots\}$$

Thus, this is a sequence of nested balls, and their intersection is empty.

One can construct a similar example with $X = (0, \infty), d(x, y) = 1 + 1/(x + y)$.

Problem 10 (extra credit).

A sequence $\{x_n\}$ in a metric space (X, d) is called a *Cauchy sequence* if $\limsup_{k \geq 1} d(x_n, x_{n+k})$ goes to zero as $n \to \infty$, or, equivalently, if for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $d(x_n, x_{n+k}) < \epsilon$ for any $n \geq N$ and $k \geq 0$.

Prove that the following sequences of rational numbers are Cauchy sequences with respect to the p-adic distance d_p introduced in Assignment 1:

a) $x_n = 1 + p + p^2 + \ldots + p^n$;

b) $y_n = (p-1) + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^n$.

Find the limits (with respect to the p-adic distance d_p) of the following sequences:

c) $(1-p)x_n$, where x_n is as in a);

d) $1 + y_n$, where y_n is as in b).

Solution: a) We have

$$|x_{n+k} - x_n| = p^{n+k} + p^{n+k-1} + \cdots + p^{n+1} = p^{n+1}(p^{k-1} + \cdots + p + 1),$$

Since $p \not| (p^{k-1} + \cdots + p + 1)$, we have $d_p(x_{n+k}, x_n) = p^{-(n+1)} \to 0$ as $n \to \infty$, for all $k \geq 1$. Hence (x_n) is Cauchy.

b)$y_n = (p-1)x_n$ so

$$\begin{aligned} |y_{n+k} - y_n| &= (p-1)(p^{n+k} + p^{n+k-1} + \cdots + p^{n+1}) = (p-1)p^{n+1}(p^{k-1} + \cdots + p + 1) \\ &= p^{n+1}(p^k - 1), \end{aligned}$$

so we have $d_p(y_{n+k}, y_n) = p^{-(n+1)} \to 0$ as $n \to \infty$, for all $k \geq 1$.

c)Let $a_n = x_n(1-p) = 1 - p^{n+1}$. Then

$$|a_n - 1| = |1 - p^{n+1} - 1| = p^{n+1},$$

so $d_p(a_n, 1) = p^{-(n+1)} \to 0$, as $n \to \infty$. Thus, $\lim_{n\to\infty} a_n = 1$.

d)Let $b_n = 1 + y_n = 1 + (p-1)x_n$. Then

$$|b_n| = |1 + y_n| = |1 + p^{n+1} - 1| = p^{n+1}.$$

Thus, $d_p(b_n, 0) = p^{-(n+1)} \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} b_n = 0$.