McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 1 — Solutions to selected Exercises

Exercise 1.

(i) Verify the identity for any two sets of complex numbers $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - b_i a_j)^2.$$

ii) Let $f(x)$ and $g(x)$ be continuous functions on $[a, b]$. Prove that

$$\left(\int_a^b f(x)g(x)dx\right)^2 = \int_a^b f(x)^2 dx \cdot \int_a^b g(x)^2 dx - \frac{1}{2}\int_a^b \int_a^b \left[f(x)g(y) - g(x)f(y)\right]^2 dxdy.$$

Solution (i) : The RHS expands to

$$\sum_{i,j=1}^{n} a_i^2 b_j^2 - \frac{1}{2}\sum_{i,j=1}^{n}(a_i^2 b_j^2 + b_i^2 a_j^2 - 2a_i b_i a_j b_j) = \sum_{i,j=1}^{n} a_i b_i a_j b_j = \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

Solution (ii) : Because $f, g \in C([a, b])$, we may very safely invoke Fubini's theorem to change the order of integration. The RHS expands to

$$\int_a^b f(x)^2 dx \cdot \int_a^b g(x)^2 dx - \frac{1}{2}\int_a^b \int_a^b \left[f(x)^2 g(y)^2 + g(x)^2 f(y)^2 - 2f(x)g(x)f(y)g(y)\right] dxdy$$

$$= \int_a^b \int_a^b \left(f(x)g(x)f(y)g(y)\right) dxdy$$

$$= \int_a^b \left(\int_a^b f(x)g(x)f(y)g(y)dx\right) dy$$

$$= \int_a^b \left(\int_a^b f(x)g(x)dx\right) f(y)g(y)dy$$

$$= \left(\int_a^b f(x)g(x)dx\right)\left(\int_a^b f(y)g(y)dy\right)$$

Exercise 2.

(i) Starting from the inequality $xy \le x^p/p + y^q/q$, where $x, y, p, q > 0$ and $1/p + 1/q = 1$, deduce *Hölder's integral inequality* for continuous functions $f(t), g(t)$ on $[a, b]$:

$$\int_a^b f(t)g(t)dt \le \left(\int_a^b |f(t)|^p dt\right)^{1/p} \left(\int_a^b |g(t)|^q dt\right)^{1/q};$$

(ii) Use (i) to prove *Minkowski's integral inequality* for continuous functions $f(t), g(t)$ on $[a, b]$ and $p \ge 1$:

$$\left(\int_a^b |f(t)+g(t)|^p dt\right)^{1/p} \le \left(\int_a^b |f(t)|^p dt\right)^{1/p} + \left(\int_a^b |g(t)|^p dt\right)^{1/p}.$$

Solution (i) : If f or g is identically 0 on $[a, b]$, then the inequality holds. So we may assume that $\|f\|_p, \|g\|_q > 0$. Take

$$x = \frac{|f(t)|}{\left(\int_a^b |f(t)|^p dt\right)^{1/p}} = \frac{|f(t)|}{\|f\|_p} \quad \text{and} \quad y = \frac{|g(t)|}{\left(\int_a^b |g(t)|^q dt\right)^{1/q}} = \frac{|g(t)|}{\|g\|_q}$$

Then

$$\int_a^b \frac{|f(t)|}{\|f\|_p}\frac{|g(t)|}{\|g\|_q}dt \le \int_a^b \left(\frac{|f(t)|^p}{p\|f\|_p^p} + \frac{|g(t)|^q}{q\|g\|_q^q}\right) dt = 1.$$

Rearrange to get

$$\int_a^b f(t)g(t)dt \le \int_a^b |f(t)g(t)|dt \le \|f\|_p\|g\|_q.$$

Young's Inequality (i) : For any $x, y, p, q > 0$, with p, q conjugate exponents, $xy \le x^p/p + y^q/q$.
Proof:

$$xy = e^{\ln xy} = e^{\frac{1}{p}\ln x^p + \frac{1}{q}\ln y^q} \le \frac{1}{p}e^{\ln x^p} + \frac{1}{q}e^{\ln y^q} = \frac{x^p}{p} + \frac{y^q}{q}$$

by convexity of the exponential function.

Solution (ii) : Minkowski's inequality holds for $p = 1$, so we may assume $p > 1$.

$$\int_a^b |f(t)+g(t)|^p dt = \int_a^b |f(t)+g(t)||f(t)+g(t)|^{p-1}dt \le \int_a^b |f(t)||f(t)+g(t)|^{p-1}dt + \int_a^b |g(t)||f(t)+g(t)|^{p-1}dt$$

$$\le \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |f(t)+g(t)|^{q(p-1)}dt\right)^{\frac{1}{q}} + \left(\int_a^b |g(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |f(t)+g(t)|^{q(p-1)}dt\right)^{\frac{1}{q}}$$

$$= \left(\left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p dt\right)^{\frac{1}{p}}\right) \left(\int_a^b |f(t)+g(t)|^{q(p-1)}dt\right)^{\frac{1}{q}}$$

Since $q(p-1) = p$, rearrange to get

$$\left(\int_a^b |f(t)+g(t)|^p dt\right)^{\frac{1}{p}} = \left(\int_a^b |f(t)+g(t)|^p dt\right)^{1-\frac{1}{q}} \le \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p dt\right)^{\frac{1}{p}}.$$

Exercise 3. Prove that the set of all points $x = (x_1, x_2, \ldots, x_k, \ldots)$ with only finitely many nonzero coordinates, each of which is a rational number, is dense in the space l_2 of sequences.

Solution: Let $x = (x_1, x_2, \ldots, x_n, \ldots) \in l_2$. Then $\sum_{i=1}^{\infty} x_i^2 < \infty$, so for any $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $\sum_{i>N} x_i^2 < \epsilon$. For each $1 \leq j \leq N$, choose a rational number y_i such that $(x_i - y_i)^2 < \epsilon/N$. Let $y = (y_1, y_2, \ldots, y_N, 0, 0, \ldots)$. Then y has only N nonzero rational coordinates, and

$$(d_2(x, y))^2 = \sum_{j=1}^{N} (x_i - y_i)^2 + \sum_{i>N} x_i^2 < \epsilon + N(\epsilon/N) = 2\epsilon.$$

Since ϵ was arbitrary, we have proved the density.

Exercise 4 (extra credit).

i) Suppose $\phi \in C([a,b])$ (which need not be differentiable) satisfies

$$\phi((x+y)/2) \leq (\phi(x)+\phi(y))/2, \qquad x,y \in [a,b].$$

Prove that for all $x,y \in [a,b]$, and for any $t \in [0,1]$, we have

$$\phi(tx+(1-t)y) \leq t\phi(x)+(1-t)\phi(y), \tag{1}$$

i.e. that ϕ is *convex* on $[a,b]$.

ii) Assume that a function ϕ (that is *not* assumed to be continuous on an *open* interval (a,b)), satisfies (1). Prove that ϕ is then actually continuous on (a,b).

iii) Prove that if $\phi \in C^2([a,b])$, and $\phi''(x) > 0, \forall x \in [a,b]$, then ϕ is convex on $[a,b]$.

iv) Prove that if $x_1, \ldots, x_n \in [a,b]$, and $t_1, \ldots, t_n > 0$ satisfy $t_1 + \ldots + t_n = 1$, and if ϕ is convex on $[a,b]$, then

$$\phi(t_1x_1 + \ldots + t_nx_n) \leq t_1\phi(x_1) + \ldots + t_n\phi(x_n).$$

Solution i):

Let $x,y \in (a,b)$ and define $f(\lambda) = \varphi((1-\lambda)x+\lambda y)$ for $0 \leq \lambda \leq 1$. Note first that $f(q) \leq (1-q)f(0)+qf(1)$ for all dyadic rationals $0 \leq q \leq 1$. To see this, suppose that the inequality holds for dyadic rationals of the form $q = \frac{k}{2^n}$ for $1 \leq n \leq N$; then if $0 \leq k < 2^N$, we have

$$\begin{aligned} f\left(\frac{2k+1}{2^{N+1}}\right) &\leq \frac{1}{2}\left(f\left(\frac{k}{2^N}\right)+f\left(\frac{k+1}{2^N}\right)\right) \\ &\leq \frac{1}{2}\left(\left(1-\frac{k}{2^N}\right)f(0)+\frac{k}{2^N}f(1)+\left(1-\frac{k+1}{2^N}\right)f(0)+\frac{k+1}{2^N}f(1)\right) \\ &= \left(1-\frac{2k+1}{2^{N+1}}\right)f(0)+\frac{2k+1}{2^{N+1}}f(1) \end{aligned}$$

Consider now the sequence of dyadic rationals obtained by taking successively accurate approximations to the binary expansion of λ. Then $\lambda_n = 2^{-n}\lfloor \lambda 2^n \rfloor$ is a sequence converging to λ, and since f is continuous the inequality holds in the limit; that is, $f(\lambda) \leq (1-\lambda)f(0)+\lambda f(1)$ for all $0 \leq \lambda \leq 1$, and hence φ is convex.

ii): We find that the definition is convexity is equivalent to requiring that for all $a < s < t < u < b$, we have

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(t)}{u-t}. \tag{2}$$

Fix $t \in (a,b)$; we shall prove that ϕ is continuous at t. Let $r(t,s_0)$ denote the ratio in the left-hand side of (2) for some fixed $a < s_0 < t$; similarly, denote $r(t,u_0)$ denote the ratio in the right-hand side of (2) for some fixed $t < u_0 < b$.

Suppose now that $s \in (s_0,t)$. Then it follows from (2) that

$$\phi(t)-r(t,u_0)(t-s) \leq \phi(s) \leq \phi(t)-r(s_0,t)(t-s),$$

i.e. the graph of ϕ lies in between two straight lines that intersect at the point $(t,\phi(t))$. The continuity as $s \to t$ from the left follows. The proof of continuity as $u \to t$ from the right follows similarly from the inequality

$$\phi(t)+r(s_0,t)(u-t) \leq \phi(u) \leq \phi(t)+r(t,u_0)(u-t),$$

that holds for $u \in (t,u_0)$.

iii): By the argument in ii), it suffices to prove (2), which for continuously differentiable functions is equivalent to saying that ϕ' is nondecreasing, and that follows from the assumption $\phi''(t) > 0, t \in [a,b]$.

iv): The proof is by induction, starting with $n = 2$ which is the assumption of continuity. The induction step is proved as follows:

$$\phi(t_1x_1+\ldots+t_nx_n+t-n+1x_{n+1}) = \phi(t_1x_1+\ldots+(t_n+t_{n+1})y) \leq t_1\phi(x_1)+\ldots+(t_n+t_{n+1})\phi(y), \quad (3)$$

where $y = (t_nx_n + t_{n+1}x_{n+1})/(t_n + t_{n+1})$ and where we have used induction hypothesis. On the other hand, by convexity

$$\phi(y) \leq \frac{t_n\phi(x_n)}{t_n + t_{n+1}} + \frac{t_{n+1}\phi(x_{n+1})}{t_n + t_{n+1}}.$$

Substituting into (3), we complete the proof.

Exercise 5. Let X be a metric space, $A \subseteq X$ a subset of X, and x a point in X. The *distance from x to A* is denoted by $d(x, A)$ and is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Prove that

i) If $x \in A$, then $d(x, A) = 0$, but not conversely;

ii) For a fixed A, $d(x, A)$ is a continuous function of x;

iii) $d(x, A) = 0$ if and only if x is a contact point of A (i.e. every neighborhood of x contains a point from A);

iv) The closure $\overline{A}$ satisfies
$$\overline{A} = A \cup \{x : d(x, A) = 0\}.$$

Solution (i) : If $x \in A$, then $0 = d(x, x) = \inf_{a \in A} d(x, a)$. Converse is not true, for example if $A = (0, 1]$, then $d(0, A) = 0$ while $0 \notin A$.

Solution (ii) : Let $x, y \in X$. Given $\epsilon > 0$, choose $a \in A$ such that $d(x, a) \leq d(x, A) + \epsilon$. By triangle inequality we have $d(y, a) \leq d(x, a) + d(x, y) \leq d(x, y) + d(x, A) + \epsilon$. Since ϵ was arbitrary, and since $d(y, A) \leq d(y, a)$, we get $d(y, A) \leq d(x, A) + d(x, y)$. Reversing the roles of x and y we get $d(x, A) \leq d(y, A) + d(x, y)$. It follows that

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

which implies continuity of $d(\cdot, A)$.

Solution (iii) : If x is a contact point of A, then for every $r > 0$, $B(x, r)$ contains a point of A, hence $\inf_{a \in A} d(x, a) < r$. Since r was arbitrary, $d(x, A) = 0$, proving the "if" part. Now, suppose a ball $B(x, r)$ doesn't contain points from A for some $r > 0$. Then $d(x, A) \geq r > 0$, finishing the proof of the "only if" part of the statement.

Solution (iv) : The set $\overline{A}$ is a union of A and the set of all limit points of A. By part (iii), $d(x, A) = 0$ for any limit point that doesn't belong to A.

Remark : To show that the function $d : X \longrightarrow [0, \infty)$, $d : x \longrightarrow d(x, A)$ was continuous, some of you came up with intriguing inequalities such as $|d(x, A) - d(y, A)| = |\inf\limits_{a \in A} d(x, a) - \inf\limits_{a \in A} d(y, a)| \leq |\inf\limits_{a \in A} (d(x, a) - d(y, a))|$. The following counterexample shows that it is wrong in general *even if y can be made arbitrarily close to x.* Consider the metric space of bounded sequences $l^\infty(\mathbb{N})$ with entries in $\mathbb{R}$ equipped with the metric $d(x, y) = \sup\limits_{i \in \mathbb{N}} |x_i - y_i|$. Let $x = (0, 0, ...)$, $y_1 = (\frac{1}{3}, 0, 0, ...)$, $y_2 = (0, (\frac{1}{3})^2, 0, ...)$, $y_3 = (0, 0, (\frac{1}{3})^3, 0, ...)$, etc . . . and $A = \{\omega \in l^\infty(\mathbb{N})$: all components of ω are 0 except exactly one which is equal to $1\}$. Then $y_n \longrightarrow x$ and $d(x, A) = 1$, $d(y_n, A) = 1 - (\frac{1}{3})^n$ so that we have $|\inf\limits_{a \in A} d(x, a) - \inf\limits_{a \in A} d(y, a)| = (\frac{1}{3})^n > 0 = |\inf\limits_{a \in A} (d(x, a) - d(y, a))|$.

Exercise 6. Let (X, d) be a metric space, and $f : X \to \mathbf{R}$ a continuous function. The *nodal set* of f, denoted by $Z(f)$, is the set $\{x \in X : f(x) = 0\}$.

i) Prove that $Z(f)$ is a closed subset of X.

Next, let A, B be two closed nonempty subsets of X, $A \cap B = \emptyset$. Let $d(x, A)$ (resp. $d(x, B)$) denote the distance from $x \in X$ to A (resp. B), defined in Exercise 5 in Assignment 1. Define a function $F : X \to \mathbf{R}$ by the formula

$$F(x) \;=\; \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Prove that

ii) F is continuous;

iii) $F(x) = 0$ iff $x \in A$, and $F(x) = 1$ iff $x \in B$.

Solution (i) : Let $x_n \in Z(f)$, and let $x_n \to y$ as $n \to \infty$. By continuity of f, $0 = f(x_n) \to f(y)$, therefore $f(y) = 0$ and so $y \in Z(f)$.

Solution (ii) and (iii) : By the results proved in Exercise 5, Assgmt 1, $d(x, A) = 0$ iff $x \in \overline{A} = A$, since A is closed, and similarly for B. It was also shown in Exercise 5, Assgmt 1, that $|d(x, A) - d(y, A)| \leq d(x, y)$. These results are used abundantly in the following demonstration. There are 3 possible different cases:

- $x \in A$: then $F(x) = 0$. Let $b = d(x, B) > 0$, and let $\epsilon < b$. Suppose that $y \in X$ is such that $d(x, y) < \epsilon$. Then $d(y, A) \leq d(x, y) < \epsilon$, and $b - \epsilon \leq d(y, B) \leq b + \epsilon$. It follows that

  $$F(y) \leq \epsilon/(b - \epsilon) \to 0 = F(x)$$

  as $\epsilon \to 0$, so F is continuous at x.

- $x \in B$: then $d(x, B) = 0$ so $F(x) = 1$. Let $a = d(x, A) > 0$. Choose $\epsilon < a$ and suppose $y \in X$ is such that $d(x, y) < \epsilon$. By an argument similar to the argument above, we find that $d(y, B) < \epsilon$ and $a - \epsilon \leq d(y, A) \leq a + \epsilon$. Accordingly,

  $$F(x) = 1 \geq F(y) = \frac{1}{1 + d(y, B)/d(y, A)} \geq \frac{1}{1 + \epsilon/(a - \epsilon)} \to 1,$$

  as $\epsilon \to 0$, proving that F is continuous at x.

- $x \notin A$ and $x \notin B$: Let $a = d(x, A) > 0$ and let $b = d(x, B) > 0$. We have $0 < F(x) = a/(a + b) < 1$. Choose $\epsilon < \min(a, b)$, and suppose $y \in X$ is such that $d(x, y) < \epsilon$. It follows that $a - \epsilon < d(y, A) < a + \epsilon$, and $b - \epsilon < d(y, B) < b + \epsilon$. Then

  $$\frac{1}{1 + (b + \epsilon)/(a - \epsilon)} \leq F(y) \leq \frac{1}{1 + (b - \epsilon)/(a + \epsilon)}.$$

  Both sides of the inequality converge to $a/(a + b)$ as $\epsilon \to 0$, proving the continuity of F at x.

Exercise 7. Let Mat_n denote the space of $n \times n$ real matrices. For $A \in \text{Mat}_n$, define the norms $||A||_1$ as follows:

$$||A||_1 = \sup_{0 \neq \mathbf{x} \in \mathbf{R}^n} \frac{||A\mathbf{x}||}{||\mathbf{x}||},$$

where $||x||$ is the usual Euclidean norm. Next define another norm $||A||_2$ by

$$||A||_2 = \max_{1 \leq i,j \leq n} |A_{ij}|.$$

Prove that

i) Prove that $||A||_{1,2}$ defines a norm on Mat_n;

ii) Prove that there exists a constant $C_n > 1$ such that $1/C_n \leq ||A||_1/||A||_2 \leq C_n$.

Solution (i) : The only nontrivial property is the triangle inequality, $||A+B||_{1,2} \leq ||A||_{1,2} + ||B||_{1,2}$; the other properties are very easy. Now,

$$||(A+B)||_1 = \sup_{||\mathbf{x}||=1} ||A\mathbf{x}+B\mathbf{x}|| \leq \sup_{||\mathbf{x}||=1} (||A\mathbf{x}||+||B\mathbf{x}||) \leq \sup_{||\mathbf{x}||=1} ||A\mathbf{x}|| + \sup_{||\mathbf{x}||=1} ||B\mathbf{x}|| = ||A||_1 + ||B||_1.$$

$$||(A+B)||_2 = \max_{i,j} |(A+B)_{ij}| \leq \max_{i,j}(|A_{ij}| + |B_{ij}|) \leq \max_{i,j} |A_{ij}| + \max_{i,j} |B_{ij}| = ||A||_2 + ||B||_2.$$

Solution (ii) : Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be an arbitrary unit vector in $(\mathbb{R}^n, ||\cdot||)$, where $||\cdot||$ stands for the Euclidean 2 norm. Then by Cauchy-Schwartz (or Hölder's inequality),

$$||A\mathbf{x}|| = \sqrt{\sum_{i=1}^n (\sum_{j=1}^n A_{ij}x_j)^2} \leq \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n A_{ij}^2\right) \left(\sum_{j=1}^n x_j^2\right)} = \sqrt{\sum_{1 \leq i,j \leq n} A_{ij}^2} \leq \sqrt{\sum_{1 \leq i,j \leq n} ||A||_2^2} = n||A||_2.$$

Denote $\mathbf{e}_j = (0, \ldots, 1, 0, \ldots, 0)$ where the 1 occurs in the j^{th} position. Then $A\mathbf{e}_j = (A_{1j}, A_{2j}, \ldots, A_{nj})$ and it follows that

$$||A||_1 = \sup_{||\mathbf{x}||=1} ||A\mathbf{x}|| \geq ||A\mathbf{e}_j|| = \sqrt{\sum_{i=1}^n A_{ij}^2} \geq |A_{ij}| \qquad \forall 1 \leq i,j \leq n$$

So $||A||_1 \geq ||A||_2$. Therefore

$$\frac{1}{n} \leq 1 \leq \frac{||A||_1}{||A||_2} \leq n.$$

which shows that $||\cdot||_1$ and $||\cdot||_2$ are equivalent norms on Mat_n.

Exercise 8 (extra credit). Let p be a prime number (a positive integer that is only divisible by 1 and itself, e.g. $p = 2, 3, 5, 7, 11$ etc). Define *p-adic distance* d_p on the set $\mathbb{Q}$ of rational numbers as follows: given $q_1, q_2 \in \mathbb{Q}$, let $|q_1 - q_2| = q \in \mathbb{Q}$. If $q_1 = q_2, q = 0$, then we set $d_p(q_1, q_2) = 0$. If $q \neq 0$, we can write q as

$$q = p^m \frac{a}{b}, \qquad where\ m \in \mathbf{Z},\ GCD(a,b) = 1,\ GCD(a,p) = GCD(b,p) = 1.$$

Here $GCD(a, b)$ is the greatest common divisor of two natural numbers a and b. Then we define the p-adic distance by

$$d_p(q_1, q_2) = p^{-m}.$$

Please, note the minus sign in the definition.

Examples: $d_2(5/2, 1/2) = 1/2$; $d_3(17, 8) = 1/9$; $d_5(4/15, 1/15) = 5$.

Prove that d_p satisfies all the properties of a distance. The only nontrivial part is the triangle inequality:

$$d_p(q_1, q_2) + d_p(q_2, q_3) \geq d_p(q_1, q_3).$$

You may use without proof all standard properties of the greatest common divisor, prime decomposition etc.

Solution : Let us introduce the p-adic norm on the vector space $\mathbb{Q}$ over itself defined by $||x||_p = p^{-m}$, for $x = p^m \cdot (a/b)$, where $GCD(a, b) = GCD(a, p) = GCD(b, p) = 1$ and $||0||_p = 0$.

- $||x||_p = 0$ iff $x = 0$.
- For $x, y \in \mathbb{Q}$, $x = p^m \frac{a}{b}, y = p^k \frac{c}{d}$, then $||x \cdot y||_p = ||p^{m+k} \frac{ac}{bd}||_p = p^{-m-k} = p^{-m}p^{-k} = ||x||_p||y||_p$
- For the triangle inequality we will show that $||x + y||_p \leq \max\{||x||_p, ||y||_p\}$. Assume without loss of generality that
$$\max\{||x||_p, ||y||_p\} = ||x||_p := p^{-m},$$
i.e. that $x = p^m(a/b), y = p^{m+k}(c/d)$, where $GCD(a, p) = 1 = GCD(b, p) = GCD(c, p) = GCD(d, p)$, and where $k \geq 0$. Then
$$x + y = p^m \frac{(p^k \cdot ad + bc)}{bd}$$
Since $GCD(p, bd) = 1$, we see that $||x+y||_p \leq p^{-m}$. The norm could be smaller, if $GCD(p, p^k ad + bc) = p$.

Hence $|| \cdot ||_p$ defines a norm on $\mathbb{Q}$ over $\mathbb{Q}$. Now if $q_1, q_2 \in \mathbb{Q}$ are distinct, let $|q_1 - q_2| = p^m \frac{a}{b}$ be the unique decomposition. Then $d_p(q_1, q_2) = p^{-m} = ||p^m \frac{a}{b}||_p = |||q_1 - q_2|||_p = ||q_1 - q_2||_p$. If $q_1 = q_2$ then $d_p(q_1, q_2) = 0 = ||0||_p = ||q_1 - q_2||_p$. So the p-adic norm induces the p-adic distance d_p.

Exercise 9 (extra credit). Denote by $\mathcal{P}$ the set of polygons in $\mathbb{R}^2$, not necessarily convex. A *polygon* P with vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is the set of points in $\mathbb{R}^2$ bounded by a simple closed curve that is a union of line segments

$$[\mathbf{x}_1, \mathbf{x}_2], [\mathbf{x}_2, \mathbf{x}_3], \ldots, [\mathbf{x}_{n-1}, \mathbf{x}_n], [\mathbf{x}_n, \mathbf{x}_1].$$

The boundary curve is denoted ∂P and is sometimes called a *polyline* or a *broken line*. We require that different line segments do not intersect except at common endpoints.

A *symmetric difference* of two sets A, B is denoted by $A \Delta B$ and is defined by

$$A \Delta B = (A \backslash B) \cup (B \backslash A),$$

where $A \backslash B = A \cap B^c$ is the set of points $\{x \in A, x \notin B\}$.

Given two polygons $P_1, P_2 \in \mathbb{R}^2$, define the distance between them by

$$d(P_1, P_2) \ = \ \mathrm{Area}(P_1 \Delta P_2).$$

Prove that d satisfies all the properties of a distance. Hint: if $X \subset Y$, then $\mathrm{Area}(X) \leq \mathrm{Area}(Y)$.

Solution:

- $d(P_1, P_2) \geq 0$ and $d(P_1, P_1) = 0 \ \forall P_1, P_2 \in \mathcal{P}$.
- $d(P_1, P_2) = \mathrm{Area}(P_1 \Delta P_2) = \mathrm{Area}(P_2 \Delta P_1) = d(P_1, P_2)$.
- As shown in the lemma below, for any sets P_1, P_2, P_3 we have

  $$(P_1 \Delta P_2) \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3).$$

  In particular the relation holds for polygons in $\mathbb{R}^2$. Taking areas, we find that

  $$\mathrm{Area}(P_1 \Delta P_2) \leq \mathrm{Area}((P_1 \Delta P_3) \cup (P_2 \Delta P_3)) \leq \mathrm{Area}(P_1 \Delta P_3) + \mathrm{Area}(P_2 \Delta P_3).$$

Lemma: For arbitrary sets P_1, P_2, P_3, $\ (P_1 \Delta P_2) \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3)$.

Proof: $P_1 \cap P_2^c = (P_1 \cap P_2^c \cap P_3) \cup (P_1 \cap P_2^c \cap P_3^c)$. The first set in parenthesis is contained in $P_2^c \cap P_3 \subset (P_2 \Delta P_3)$, while the second set in parenthesis is contained in $P_1 \cap P_3^c \subset (P_1 \Delta P_3)$. So, $P_1 \cap P_2^c \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3)$. Reversing the roles of P_1 and P_2, we see that $P_2 \cap P_1^c \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3)$. Therefore $(P_1 \Delta P_2) = (P_1 \cap P_2^c) \cup (P_2 \cap P_1^c) \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3)$.