McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 1 — Solutions to selected problems

Problem 3. Prove that the set of all points $x = (x_1, x_2, \ldots, x_k, \ldots)$ with only finitely many nonzero coordinates, each of which is a rational number, is dense in the space l_2 of sequences.

Solution: Let $x = (x_1, x_2, \ldots, x_n, \ldots) \in l_2$. Then $\sum_{i=1}^{\infty} x_i^2 < \infty$, so for any $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $\sum_{i>N} x_i^2 < \epsilon$. For each $1 \leq j \leq N$, choose a rational number y_i such that $(x_i - y_i)^2 < \epsilon/N$. Let $y = (y_1, y_2, \ldots, y_N, 0, 0, \ldots)$. Then y has only N nonzero rational coordinates, and

$$(d_2(x,y))^2 = \sum_{j=1}^{N}(x_i - y_i)^2 + \sum_{i>N} x_i^2 < \epsilon + N(\epsilon/N) = 2\epsilon.$$

Since ϵ was arbitrary, we have proved the density.

Problem 4 (extra credit).

i) Suppose $\phi \in C([a,b])$ (which need not be differentiable) satisfies

$$\phi((x+y)/2) \leq (\phi(x) + \phi(y))/2, \qquad x, y \in [a,b].$$

Prove that for all $x, y \in [a,b]$, and for any $t \in [0,1]$, we have

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y), \tag{1}$$

i.e. that ϕ is *convex* on $[a,b]$.

ii) Assume that a function ϕ (that is *not* assumed to be continuous on an *open* interval (a,b)), satisfies (1). Prove that ϕ is then actually continuous on (a,b).

iii) Prove that if $\phi \in C^2([a,b])$, and $\phi''(x) > 0, \forall x \in [a,b]$, then ϕ is convex on $[a,b]$.

iv) Prove that if $x_1, \ldots, x_n \in [a,b]$, and $t_1, \ldots, t_n > 0$ satisfy $t_1 + \ldots + t_n = 1$, and if ϕ is convex on $[a,b]$, then

$$\phi(t_1 x_1 + \ldots + t_n x_n) \leq t_1\phi(x_1) + \ldots + t_n\phi(x_n).$$

Solution i):

Let $x, y \in (a,b)$ and define $f(\lambda) = \varphi((1-\lambda)x + \lambda y)$ for $0 \leq \lambda \leq 1$. Note first that $f(q) \leq (1-q)f(0) + qf(1)$ for all dyadic rationals $0 \leq q \leq 1$. To see this, suppose that the inequality holds for dyadic rationals of the form $q = \frac{k}{2^n}$ for $1 \leq n \leq N$; then if $0 \leq k < 2^N$, we have

$$\begin{aligned}
f\left(\frac{2k+1}{2^{N+1}}\right) &\leq \frac{1}{2}\left(f\left(\frac{k}{2^N}\right) + f\left(\frac{k+1}{2^N}\right)\right) \\
&\leq \frac{1}{2}\left(\left(1 - \frac{k}{2^N}\right)f(0) + \frac{k}{2^N}f(1) + \left(1 - \frac{k+1}{2^N}\right)f(0) + \frac{k+1}{2^N}f(1)\right) \\
&= \left(1 - \frac{2k+1}{2^{N+1}}\right)f(0) + \frac{2k+1}{2^{N+1}}f(1)
\end{aligned}$$

Consider now the sequence of dyadic rationals obtained by taking successively accurate approximations to the binary expansion of λ. Then $\lambda_n = 2^{-n}\lfloor \lambda 2^n \rfloor$ is a sequence converging to λ, and since f

is continuous the inequality holds in the limit; that is, $f(\lambda) \leq (1-\lambda)f(0) + \lambda f(1)$ for all $0 \leq \lambda \leq 1$, and hence φ is convex.
ii): We find that the definition is convexity is equivalent to requiring that for all $a < s < t < u < b$, we have

$$\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t}. \tag{2}$$

Fix $t \in (a, b)$; we shall prove that ϕ is continuous at t. Let $r(t, s_0)$ denote the ratio in the left-hand side of (2) for some fixed $a < s_0 < t$; similarly, denote $r(t, u_0)$ denote the ratio in the right-hand side of (2) for some fixed $t < u_0 < b$.

Suppose now that $s \in (s_0, t)$. Then it follows from (2) that

$$\phi(t) - r(t, u_0)(t - s) \leq \phi(s) \leq \phi(t) - r(s_0, t)(t - s),$$

i.e. the graph of ϕ lies in between two straight lines that intersect at the point $(t, \phi(t))$. The continuity as $s \to t$ from the left follows. The proof of continuity as $u \to t$ from the right follows similarly from the inequality

$$\phi(t) + r(s_0, t)(u - t) \leq \phi(u) \leq \phi(t) + r(t, u_0)(u - t),$$

that holds for $u \in (t, u_0)$.
iii): By the argument in ii), it suffices to prove (2), which for continuously differentiable functions is equivalent to saying that ϕ' is nondecreasing, and that follows from the assumption $\phi''(t) > 0, t \in [a, b]$.
iv): The proof is by induction, starting with $n = 2$ which is the assumption of continuity. The induction step is proved as follows:

$$\phi(t_1x_1 + \ldots + t_nx_n + t - n + 1x_{n+1}) = \phi(t_1x_1 + \ldots + (t_n + t_{n+1})y) \leq t_1\phi(x_1) + \ldots + (t_n + t_{n+1})\phi(y), \tag{3}$$

where $y = (t_nx_n + t_{n+1}x_{n+1})/(t_n + t_{n+1})$ and where we have used induction hypothesis. On the other hand, by convexity

$$\phi(y) \leq \frac{t_n\phi(x_n)}{t_n + t_{n+1}} + \frac{t_{n+1}\phi(x_{n+1})}{t_n + t_{n+1}}.$$

Substituting into (3), we complete the proof.
Problem 5. Let X be a metric space, $A \subseteq X$ a subset of X, and x a point in X. The *distance from x to A* is denoted by $d(x, A)$ and is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Prove that

i) If $x \in A$, then $d(x, A) = 0$, but not conversely;

ii) For a fixed A, $d(x, A)$ is a continuous function of x;

iii) $d(x, A) = 0$ if and only if x is a contact point of A (i.e. every neighborhood of x contains a point from A);

iv) The closure $\overline{A}$ satisfies

$$\overline{A} = A \cup \{x : d(x, A) = 0\}.$$

Solution: (i) If $x \in A$, then $0 = d(x,x) = \inf_{a\in A} d(x,a)$. Converse is not true, for example if $A = (0,1]$, then $d(0,A) = 0$ while $0 \notin A$.

(ii) Let $x, y \in X$. Given $\epsilon > 0$, choose $a \in A$ such that $d(x,a) \le d(x,A) + \epsilon$. By triangle inequality we have $d(y,a) \le d(x,a) + d(x,y) \le d(x,y) + d(x,A) + \epsilon$. Since ϵ was arbitrary, and since $d(y,A) \le d(y,a)$, we get $d(y,A) \le d(x,A) + d(x,y)$. Reversing the roles of x and y we get $d(x,A) \le d(y,A) + d(x,y)$. It follows that

$$|d(x,A) - d(y,A)| \le d(x,y).$$

This proves the continuity.

(iii) If x is a contact point of A, then for every $r > 0$, $B(x,r)$ contains a point of A, hence $\inf_{a\in A} d(x,a) < r$. Since r was arbitrary, $d(x,A) = 0$, proving the "if" part. Now, suppose a ball $B(x,r)$ doesn't contain points from A for some $r > 0$. Then $d(x,A) \ge r > 0$, finishing the proof of the "only if" part of the statement.

(iv) The set $\overline{A}$ is a union of A and the set of all limit points of A. By part (iii), $d(x,A) = 0$ for any limit point that doesn't belong to A.

Problem 6. Let (X,d) be a metric space, and $f : X \to \mathbf{R}$ a continuous function. The *nodal set* of f, denoted by $Z(f)$, is the set $\{x \in X : f(x) = 0\}$.

i) Prove that $Z(f)$ is a closed subset of X.

Next, let A, B be two closed nonempty subsets of X, $A \cap B = \emptyset$. Let $d(x,A)$ (resp. $d(x,B)$) denote the distance from $x \in X$ to A (resp. B), defined in Problem 5 in Assignment 1. Define a function $F : X \to \mathbf{R}$ by the formula

$$F(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}.$$

Prove that

ii) F is continuous;

iii) $F(x) = 0$ iff $x \in A$, and $F(x) = 1$ iff $x \in B$.

Solution: (i) Let $x_n \in Z(f)$, and let $x_n \to y$ as $n \to \infty$. By continuity of f, $0 = f(x_n) \to f(y)$, therefore $f(y) = 0$ and so $y \in Z(f)$, QED.

(ii) and (iii) By the results proved in Problem 5, Assgmt 1, $d(x,A) = 0$ iff $x \in \overline{A} = A$, since A is closed, and similarly for B. It was also shown in Problem 5, Assgmt 1, that $|d(x,A) - d(y,A)| \le d(x,y)$. Now, if $x \in A$, we have $F(x) = 0$. Let $b = d(x,B) > 0$, and let $\epsilon < b$. Suppose that $d(x,y) < \epsilon$. Then $d(y,A) \le d(x,y) < \epsilon$, and $b - \epsilon \le d(y,B) \le b + \epsilon$. It follows that

$$F(y) \le \epsilon/(b-\epsilon) \to 0 = F(x)$$

, as $\epsilon \to 0$, so F is continuous at x.

Next, suppose that $x \in B$. Then $d(x,B) = 0$ so $F(x) = 1$. Let $a = d(x,A) > 0$. Also, let $\epsilon < a$ and $d(x,y) < \epsilon$ for some $y \in X$. By an argument similar to the argument above, we find that $d(y,B) < \epsilon$ and $a - \epsilon \le d(y,A) \le a + \epsilon$. Accordingly,

$$F(x) = 1 \ge F(y) = \frac{1}{(1 + d(x,B)/d(x,A))} \ge \frac{1}{(1 + \epsilon/(a-\epsilon))} \to 1,$$

as $\epsilon \to 0$, proving that F is continuous at x.

Finally, suppose that $x \notin A$ and $x \notin B$. Let $a = d(x, A) > 0$ and let $b = d(x, B) > 0$. Finally, let $\epsilon < \min(a, b)$, and let $d(x, y) < \epsilon$. It follows that $a - \epsilon < d(y, A) < a + \epsilon$, and $b - \epsilon < d(y, B) < b + \epsilon$. We have $0 < F(x) = a/(a + b) < 1$. Also,

$$\frac{1}{[1 + (b + \epsilon)/(a - \epsilon)]} \leq F(y) \leq \frac{1}{[1 + (b - \epsilon)/(a + \epsilon)]}.$$

Both sides of the inequality converge to $a/(a + b)$ as $\epsilon \to 0$, proving the continuity of F at x. This finishes the proof.

Problem 7:

Let Mat_n denote the space of $n \times n$ real matrices. For $A \in \mathrm{Mat}_n$, define the norms $||A||_1$ as follows:

$$||A||_1 = \sup_{0 \neq \mathbf{x} \in \mathbf{R}^n} \frac{||A\mathbf{x}||}{||\mathbf{x}||},$$

where $||x||$ is the usual Euclidean norm. Next define another norm $||A||_2$ by

$$||A||_2 = \max_{i,j} |A_{ij}|.$$

Prove that

i) Prove that $||A||_{1,2}$ defines a norm on Mat_n;

ii) Prove that there exists a constant $C_n > 1$ such that $1/C_n \leq ||A||_1/||A||_2 \leq C_n$.

Solution: (i) The only nontrivial property is the equivalent of the triangle inequality, $||A + B|| \leq ||A|| + ||B||$; the other properties are very easy. Now,

$$||(A + B)||_1 = \max_{||\mathbf{x}||=1} ||A\mathbf{x} + B\mathbf{x}|| \leq \max_{||\mathbf{x}||=1} ||A\mathbf{x}|| + \max_{||\mathbf{x}||=1} ||B\mathbf{x}|| = ||A||_1 + ||B||_1.$$

Also,

$$||(A + B)||_2 = \max_{i,j} |(A + B)_{ij}| \leq \max_{i,j} |A_{ij}| + \max_{i,j} |B_{ij}| = ||A||_2 + ||B||_2.$$

The proof is finished.

(ii) To compute $||A||_1$, it suffices (by scaling) to take $||\mathbf{x}|| = 1$. Let $||A||_2 = a$ be the largest (in absolute value) matrix element. By conjugating the matrix, changing its sign, and re-labeling the coordinates, we can assume without loss of generality that (one of) the largest matrix element(s) is $A_{11} > 0$. First, we would like to show that all the coordinates of $A\mathbf{x}$ have absolute value less than or equal to $a\sqrt{n}$. Indeed, let A_j be the j-th row of A. Then $(A\mathbf{x})_j = (A_j, \mathbf{x})$. Now, by Cauchy-Schwartz inequality,

$$|(A_j, \mathbf{x})| \leq ||A_j|| \cdot ||\mathbf{x}|| \leq a\sqrt{n} \cdot 1 = a\sqrt{n}.$$

It follows that $||A\mathbf{x}|| \leq a\sqrt{n} \cdot \sqrt{n} = an$. Accordingly, $||A||_1 \leq ||A||_2 \cdot n$.

Next, choose $\mathbf{x} = e_1 = (1, 0, 0, \ldots, 0)$. Then $A\mathbf{x} = (a, 0, 0, \ldots, 0)$. It follows that

$$||A||_1 = \sup_{||\mathbf{x}||=1} ||A\mathbf{x}|| \geq ||Ae_1|| = a = ||A||_2,$$

so

$$1 \leq \frac{||A||_1}{||A||_2} \leq n.$$

Problem 8 (extra credit). Let p be a prime number (a positive integer that is only divisible by 1 and itself, e.g. $p = 2, 3, 5, 7, 11$ etc). Define *p-adic distance* d_p on the set $\mathbf{Q}$ of rational numbers as follows: given $q_1 q_2 \in \mathbf{Q}$, let $|q_1 - q_2| = q \in \mathbf{Q}$. If $q_1 = q_2, q = 0$, then we set $d_p(q_1, q_2) = 0$. If $q \neq 0$, we can write q as

$$q = p^m \, \frac{a}{b}, \qquad where\ m \in \mathbf{Z},\ GCD(a, b) = 1,\ GCD(a, p) = GCD(b, p) = 1.$$

Here $GCD(a, b)$ is the greatest common divisor of two natural numbers a and b. Then we define the p-adic distance by

$$d_p(q_1, q_2) = p^{-m}.$$

Please, note the minus sign in the definition.

Examples: $d_2(5/2, 1/2) = 1/2$; $d_3(17, 8) = 1/9$; $d_5(4/15, 1/15) = 5$.

Prove that d_p satisfies all the properties of a distance. The only nontrivial part is the triangle inequality:

$$d_p(q_1, q_2) + d_p(q_2, q_3) \geq d_p(q_1, q_3).$$

You may use without proof all standard properties of the greatest common divisor, prime decomposition etc.

We define the p-adic norm by $||x||_p = p^{-m}$, for $x = p^m \cdot (a/b)$, where $GCD(a, p) = 1 = GCD(b, p)$. It suffices to show that $||x+y||_p \leq ||x||_p + ||y||_p$. In fact, we shall see that $||x+y||_p \leq \max\{||x||_p, ||y||_p\}$, implying the previous inequality.

Assume without loss of generality that

$$\max\{||x||_p, ||y||_p\} = ||x||_p := p^{-m},$$

i.e. that $x = p^m(a/b), y = p^{m+k}(c/d)$, where $GCD(a, p) = 1 = GCD(b, p) = GCD(c, p) = GCD(d, p)$, and where $k \geq 0$. Then

$$x + y = p^m \frac{(p^k \cdot ad + bc)}{bd}$$

Since $GCD(p, bd) = 1$, we see that $||x + y||_p \leq p^{-m}$. The norm could be smaller, if $GCD(p, p^k ad + bc) = p$.

QED

Problem 9 (extra credit).

Denote by $\mathcal{P}$ the set of polygons in $\mathbf{R}^2$, not necessarily convex. A *polygon* P with vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is the set of points in $\mathbf{R}^2$ bounded by a simple closed curve that is a union of line segments

$$[\mathbf{x}_1, \mathbf{x}_2], [\mathbf{x}_2, \mathbf{x}_3], \ldots, [\mathbf{x}_{n-1}, \mathbf{x}_n], [\mathbf{x}_n, \mathbf{x}_1].$$

The boundary curve is denoted ∂P and is sometimes called a *polyline* or a *broken line*. We require that different line segments do not intersect except at common endpoints.

A *symmetric difference* of two sets A, B is denoted by $A \Delta B$ and is defined by

$$A \Delta B = (A \backslash B) \cup (B \backslash A),$$

where $A \backslash B = A \cap B^c$ is the set of points $\{x \in A, x \notin B\}$.

Given two polygons $P_1, P_2 \in \mathbf{R}^2$, define the distance between them by

$$d(P_1, P_2) \;=\; \text{Area}(P_1 \Delta P_2).$$

Prove that d satisfies all the properties of a distance. Hint: if $X \subset Y$, then $\text{Area}(X) \leq \text{Area}(Y)$.

Solution: Denote by $\mathcal{P}$ the set of polygons in $\mathbf{R}^2$, not necessarily convex. A *polygon* P with vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is the set of points in $\mathbf{R}^2$ bounded by a simple closed curve that is a union of line segments

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A *symmetric difference* of two sets A, B is denoted by $A \Delta B$ and is defined by

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Given two polygons $P_1, P_2 \in \mathbf{R}^2$, define the distance between them by

$$d(P_1, P_2) \;=\; \text{Area}(P_1 \Delta P_2).$$

Prove that d satisfies all the properties of a distance. Hint: if $X \subset Y$, then $\text{Area}(X) \le \text{Area}(Y)$.
Solution: it is easy to see that for any three polygons (or, indeed, sets!) P_1, P_2, P_3 we have

$$(P_1 \Delta P_2) \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3). \tag{4}$$

Indeed, $P_1 \cap P_2^c = (P_1 \cap P_2^c \cap P_3) \cup (P_1 \cap P_2^c \cap P_3^c)$. Now, the first set is contained in $P_2^c \cap P_3 \subset (P_2 \Delta P_3)$, while the second set is contained in $P_1 \cap P_3^c \subset (P_1 \Delta P_3)$. So, $P_1 \cap P_2^c \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3)$. Reversing the roles of P_1 and P_2, we see that $P_2 \cap P_1^c \subset (P_1 \Delta P_3) \cup (P_2 \Delta P_3)$. But $(P_1 \Delta P_2) = (P_1 \cap P_2^c) \cup (P_2 \cap P_1^c)$, and both sets are contained in the RHS of (4), finishing the proof.

Taking areas in (4), we find that

$$\text{Area}(P_1 \Delta P_2) \le \text{Area}((P_1 \Delta P_3) \cup (P_2 \Delta P_3)) \le \text{Area}(P_1 \Delta P_3) + \text{Area}(P_2 \Delta P_3),$$

proving the triangle inequality. The other two properties are obviously satisfied.