McGill University
Math 354: Honors Analysis 3

MIDTERM October 31, 2012

Do all the problems.

Problem 1 (6 points). Let $d(x, y) = |\arctan x - \arctan y|$.

a) (3 points) Prove that $d(x, y)$ defines a distance in $\mathbf{R}$.

b) (3 points) Prove that $\mathbf{R}$ is not complete with respect to the distance $d(x, y)$.

You may use all the standard properties of arctan : continuity, differentiability, monotonicity etc.

Problem 2 (8 points).

a) (2 points) State the *Contraction Mapping* theorem.

b) (6 points) Let Y be the set of continuous functions on $[0, 1]$ that take values in $[0, 1]$, with the uniform distance. Prove that Y is complete. Also, prove that the mapping A defined on Y by the formula $[Af](x) = [(f(x))^3 + x + 2]/4$ is a contraction mapping of Y into itself. Conclude that there exists a unique function $f : [0, 1] \to [0, 1]$ satisfying $f(x) = [(f(x))^3 + x + 2]/4$.

Problem 3 (9 points). Let $1 \le p < \infty$. Define the map Φ on the space l_p of real sequences by the formula

$$\Phi((x_1, x_2, \ldots, x_n, \ldots) = (x_1^2, x_2^2, \ldots, x_n^2, \ldots).$$

a) (3 points) Show that Φ is continuous everywhere in l_p.

b) (3 points) Let $p = 1$. Let A be the cylinder $\{x \in l_1 : |x_k| \le 1/k, \forall k \ge 0\}$. Is A compact? You can use any result proved in class or in a homework.

c) (3 points) Find the set $\Phi(A)$ and determine whether it is compact.

Problem 4 (7 points).

a) (2 points) State Arzela-Ascoli theorem.

Determine whether the following families of functions have compact closures in $C([0, 1])$ (with the distance induced by the norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$:

b) (2 points) $f_n(x) = e^{x^2 \sin(n)}$;

c) (3 points) the set of all quadratic polynomials $\{p(x) = x^2 + bx + c\}$ satisfying $|p(0)| \le 10$ and $|p'(1)| \le 8$.