Math 320a, Differential Geometry, D. Jakobson, Fall 2003

Finding curvature and torsion for curves not parametrized by arclength

Solution to Problem 12abc, p. 25 in do Carmo's book.

Let $\alpha(t) = (x(t), y(t), z(t)$ be our curve parametrized by t, and let $\alpha', \alpha'', \alpha'''$ denote its derivatives with respect to t. We denote the arclength by s, the curvature by κ, and the torsion by τ. We denote the unit tangent vector to α by $\overline{T}$, the unit normal vector by $\overline{N}$, and the unit binormal vector by $\overline{B}$.

Problem 12a. By definition of $s = s(t)$, we have $ds/dt = |\alpha'(t)|$. By the inverse function theorem, $dt/ds = 1/|\alpha'(t)|$. Now,

$$\frac{d^2t}{ds^2} = \frac{d}{ds}\left(\frac{dt}{ds}\right) = \frac{d/dt(dt/ds)}{ds/dt} = \frac{d/dt(|\alpha'(t)|^{-1})}{|\alpha'(t)|} = \frac{-d/dt(|\alpha'(t)|)}{|\alpha'(t)|^3} =$$
$$\frac{-1}{|\alpha'(t)|^3}\frac{d}{dt}\left[(\alpha'\cdot\alpha')^{1/2}\right] = \frac{-2\alpha'\cdot\alpha''}{|\alpha'|^3\cdot 2|\alpha'|} = \frac{-\alpha'\cdot\alpha''}{|\alpha'|^4}$$

This finishes the proof.

Problem 12b. We have $\alpha' = |\alpha'|\overline{T}$. Now,

$$\alpha'' = (d/dt)\alpha' = \frac{(d/ds)\alpha'}{dt/ds} = |\alpha'|\frac{d}{ds}(|\alpha'|\overline{T}).$$

We next evaluate

$$\frac{d}{ds}(|\alpha'|\overline{T}) = \frac{d}{ds}|\alpha'|\cdot\overline{T} + |\alpha'|\frac{d}{ds}\overline{T} = \frac{(d/dt)|\alpha'|}{ds/dt}\overline{T} + |\alpha'|\kappa\overline{N} = \frac{\alpha'\cdot\alpha''}{|\alpha'|^2}\overline{T} + |\alpha'|\kappa\overline{N},$$

where we have used Frenet's formula $(d/ds)\overline{T} = \kappa\overline{N}$, and the calculation of $(d/dt)|\alpha'|$ from Problem 12a. Accordingly,

$$\alpha'' = \frac{\alpha'\cdot\alpha''}{|\alpha'|}\overline{T} + |\alpha'|^2\kappa\overline{N},$$

and

$$\alpha'\wedge\alpha'' = |\alpha'|\overline{T}\wedge\left(\frac{\alpha'\cdot\alpha''}{|\alpha'|}\overline{T} + |\alpha'|^2\kappa\overline{N}\right) = |\alpha'|^3\kappa\overline{T}\wedge\overline{N} = |\alpha'|^3\kappa\overline{B}.$$

Computing norms, we find that

$$|\alpha'\wedge\alpha''| = \kappa|\alpha'|^3,$$

which finishes the proof.

Problem 12c. We have $\alpha'' = (\alpha'\cdot\alpha''/|\alpha'|)\overline{T} + |\alpha'|^2\kappa\overline{N}$ and $\alpha'\wedge\alpha'' = |\alpha'|^3\kappa\overline{B}$. We want to show that the torsion

$$\tau = \frac{-(\alpha'\wedge\alpha'')\cdot\alpha'''}{|\alpha'\wedge\alpha''|^2} = \frac{-(\alpha'\wedge\alpha'')\cdot\alpha'''}{\kappa^2|\alpha'|^6}. \tag{1}$$

To simplify the calculations, we remark that since $\alpha'\wedge\alpha''$ is proportional to $\overline{B}$, so we only need to compute the $\overline{B}$-component in the expansion of the vector α''' in the orthonormal basis $\{\overline{T}, \overline{N}, \overline{B}\}$. Accordingly, we write

$$\alpha''' = \frac{d}{dt}\alpha'' = \frac{d}{dt}\left(\frac{\alpha'\cdot\alpha''}{|\alpha'|}\right)\overline{T} + \frac{\alpha'\cdot\alpha''}{|\alpha'|}\cdot\frac{d}{dt}(\overline{T}) + \frac{d}{dt}(|\alpha'|^2\kappa)\overline{N} + |\alpha'|^2\kappa\cdot\frac{d}{dt}(\overline{N}).$$

The last expression can rewritten as

$$U_1 + \left(\frac{\alpha' \cdot \alpha''}{|\alpha'|}\right) \frac{(d/ds)(\overline{T})}{dt/ds} + U_2 + |\alpha'|^2 \kappa \frac{(d/ds)(\overline{N})}{dt/ds} =$$

$$U_1 + U_2 + (\alpha' \cdot \alpha'')\kappa\overline{N} + |\alpha'|^3\kappa(-\kappa\overline{T} - \tau\overline{B}) = U_1 + U_2 + U_3 - |\alpha'|^3\kappa\tau\overline{B},$$

where the vectors U_1, U_2, U_3 are orthogonal to B, and where we have used Frenet's formula $(d/ds)\overline{N} = -\kappa\overline{T} - \tau\overline{B}$. Accordingly,

$$(\alpha' \wedge \alpha'') \cdot \alpha''' = -\tau(|\alpha'|^3\kappa)^2\overline{B} \cdot \overline{B} = -\tau|\alpha'|^6\kappa^2.$$

This finishes the proof of (1).