Ordinary Differential Equations

Solutions Set assignment 7

$MATH-261B$

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1 Problem 1

Show that $J_{1/2}(x)$ is a constant multiple of $x^{-1/2}\sin(x)$.

To do this, you have to use the 2 following recurrence relations for Bessel functions:

$$\frac{d}{dx}\Big[x^{\nu}J_{\nu}(x)\Big] = x^{\nu}J_{\nu-1}(x) \tag{1}$$

$$\frac{d}{dx}\Big[x^{-\nu}J_{\nu}(x)\Big] = -x^{-\nu}J_{\nu+1}(x) \tag{2}$$

Let's take the derivative of (1) with $\nu = 1/2$ and plug (2) with $\nu = -1/2$ to get:

$$\begin{aligned}
\frac{d^2}{dx^2}\Big[x^{1/2}J_{1/2}(x)\Big] &= \frac{d}{dx}\Big[x^{1/2}J_{-1/2}(x)\Big] \\
\frac{d^2}{dx^2}\Big[x^{1/2}J_{1/2}(x)\Big] &= -x^{1/2}J_{1/2}(x) \\
\frac{d^2}{dx^2}\Big[x^{1/2}J_{1/2}(x)\Big] &+ x^{1/2}J_{1/2}(x) = 0
\end{aligned}$$

Let $z(x) = x^{1/2}J_{1/2}(x)$ and solve the ODE

$$\begin{aligned}
\frac{d^2}{dx^2}[z(x)] + z(x) &= 0 \\
z(x) = A\sin(x) &+ B\cos(x)
\end{aligned}$$

Since $z(0) = 0$, we get,

$$\begin{aligned}
z(x) &= A\sin(x) \\
x^{1/2}J_{1/2}(x) &= A\sin(x) \\
J_{1/2}(x) &\propto x^{-1/2}\sin(x)
\end{aligned}$$

2 Problem 2

Determine the Laplace transforms of the following functions:
a)

$$f(t) = \begin{cases} e^{2t} & 0 < t < 3 \\ t^2 - 4 & t > 3 \end{cases}$$

By the definition of a Laplace transform,

$$\begin{aligned} F(s) &= \int_0^3 e^{-st}e^{2t}dt + \int_3^\infty e^{-st}(t^2-4)dt \\ &= \int_0^3 e^{(2-s)t}dt + \lim_{N\to\infty}\Big(\int_3^N t^2e^{-st}dt - 4\int_3^N e^{-st}dt\Big) \end{aligned}$$

To evaluate the second integral, use integration by parts twice. We then obtain,

$$\begin{aligned} F(s) = \frac{e^{(2-s)3}-1}{2-s} + \lim_{N\to\infty}\Bigg(\frac{-N^2e^{-sN}}{s} + \frac{9e^{-3s}}{s} + \frac{2}{s}\Bigg(\frac{-Ne^{-sN}}{s} + \frac{3e^{-3s}}{s} \\ -\frac{1}{s^2}\Big(e^{-sN} - e^{-3s}\Big)\Bigg) + \frac{4}{s}\Big(e^{-sN} - e^{-3S}\Big)\Bigg) \end{aligned}$$

The limits go to zero since e^t grows faster than any power of t as it goes to infinity.

$$\begin{aligned} F(s) &= \frac{e^{3(2-s)}-1}{2-s} + \frac{9e^{-3s}}{s} + \frac{6e^{-3s}}{s^2} + \frac{2e^{-3s}}{s^3} - \frac{4e^{-3s}}{s} \\ &= \frac{e^{3(2-s)}-1}{2-s} + \frac{(5s^2+6s+2)e^{-3s}}{s^3} \end{aligned}$$

b)

$$F(s) = \mathcal{L}\{f(t)\}$$

By linearity of the Laplace transform,

$$\begin{aligned} F(s) &= \mathcal{L}\{t^5 + e^{-2t}\cos(\sqrt{3}t) - t^2e^{-2t}\} \\ &= \mathcal{L}\{t^5\} + \mathcal{L}\{e^{-2t}\cos(\sqrt{3}t)\} - \mathcal{L}\{t^2e^{-2t}\} \end{aligned}$$

Looking at a table of Laplace transforms, we get,

$$\begin{aligned} F(s) &= \frac{5!}{s^6} + \frac{s+2}{(s+2)^2+3} - \frac{2!}{(s+2)^3} \\ &= \frac{120}{s^6} + \frac{s+2}{(s+2)^2+3} - \frac{2}{(s+2)^3} \end{aligned}$$

c)

$$F(s) = \mathcal{L}\{f(t)\}$$

By linearity of the Laplace transform,

$$\begin{aligned} F(s) &= \mathcal{L}\{(t-2)^4 + t\sin^2(t) + t^2e^{3t}\} \\ &= \mathcal{L}\{(t-2)^4\} + \mathcal{L}\{t\sin^2(t)\} + \mathcal{L}\{t^2e^{3t}\} \end{aligned}$$

Using trigonometric identities, we obtain,

$$\begin{aligned} F(s) &= \mathcal{L}\{t^4 - 8t^3 + 24t^2 - 32t + 16\} + \mathcal{L}\left\{\frac{t - t\cos(2t)}{2}\right\} + \mathcal{L}\{t^2 e^{3t}\} \\ &= \frac{4!}{s^5} - 8\frac{3!}{s^4} + 24\frac{2!}{s^3} - 32\frac{1!}{s^2} + \frac{16}{s} + \frac{1!}{2s^2} - \frac{1}{2}\frac{s^2-4}{(s^2+4)^2} + \frac{2!}{(s-3)^3} \\ &= \frac{24}{s^5} - \frac{48}{s^4} + \frac{48}{s^3} - \frac{63}{2s^2} + \frac{16}{s} + \frac{2}{(s-3)^3} - \frac{1}{2}\frac{s^2-4}{(s^2+4)^2} \end{aligned}$$

d)

$$F(s) = \mathcal{L}\{f(t)\}$$

By linearity of the Laplace transform,

$$\begin{aligned} F(s) &= \mathcal{L}\{\sin(3t)\cos(3t) + t^3\cos(bt)\} \\ &= \mathcal{L}\{\sin(3t)\cos(3t)\} + \mathcal{L}\{t^3\cos(bt)\} \end{aligned}$$

Using trigonometric identities, we obtain,

$$\begin{aligned} F(s) &= \mathcal{L}\left\{\frac{\sin(6t)}{2}\right\} + (-1)^3\frac{d^3}{ds^3}\left(\frac{s}{s^2+b^2}\right) \\ &= \frac{1}{2}\frac{6}{s^2+36} - \frac{d^3}{ds^3}\left(\frac{s}{s^2+b^2}\right) \end{aligned}$$

Computing the derivatives, we get,

$$\begin{aligned} \frac{d}{ds}\left(\frac{s}{s^2+b^2}\right) &= \frac{b^2-s^2}{(s^2+b^2)^2} \\ \frac{d^2}{ds^2}\left(\frac{b^2-s^2}{(s^2+b^2)^2}\right) &= \frac{2s(s^2-3b^2)}{(s^2+b^2)^3} \\ \frac{d^3}{ds^3}\left(\frac{2s(s^2-3b^2)}{(s^2+b^2)^3}\right) &= \frac{-6(s^4-6s^2b^2+b^4)}{(s^2+b^2)^4} \end{aligned}$$

Finally,

$$F(s) = \frac{3}{s^2+36} + \left(\frac{6(s^4-6s^2b^2+b^4)}{(s^2+b^2)^4}\right)$$

3 Problem 3

Determine the Inverse Laplace transforms of the following functions:

a)

$$\begin{aligned} F(s) &= \frac{1}{s^3(s-1)^2(s+2)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{s+2} + \frac{F}{s^3} \end{aligned}$$

Putting everything on the same denominator, we obtain,

$$\begin{aligned} As^2(s-1)^2(s+2)+Bs(s-1)^2(s+2)+Cs^3(s-1)(s+2) & \\ +Ds^3(s+2)+Es^3(s-1)^2+F(s-1)^2(s+2) &= 1 \\ (A+C+D)s^5+(B+C+D-2E)s^4+(-3A-2C+2D+E+F)s^3 & \\ +(2A-3B)s^2+(2B-3F)s+2F &= 1 \end{aligned}$$

We then obtain the following system of equations,

$$\begin{aligned} A+C+D &= 0 \\ B+C+D-2E &= 0 \\ -3A-2C+2D+E+F &= 0 \\ 2A-3B &= 0 \\ 2B-3F &= 0 \\ 2F &= 1 \end{aligned}$$

Solving this by matrix method to get:

$$\begin{aligned} A &= \frac{9}{8} \\ B &= \frac{3}{4} \\ C &= \frac{-10}{9} \\ D &= \frac{1}{3} \\ E &= \frac{-1}{72} \\ F &= \frac{1}{2} \end{aligned}$$

Finally,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{9}{8s}+\frac{3}{4s^2}+\frac{1}{2s^3}-\frac{10}{9(s-1)}+\frac{1}{3(s-1)^2}-\frac{1}{72(s+2)}\right\} \\ f(t) &= \frac{9}{8}+\frac{3t}{4}+\frac{t^2}{4}-\frac{10e^t}{9}+\frac{te^t}{3}-\frac{e^{-2t}}{72} \end{aligned}$$

b)

$$F(s) = \frac{7s^2-41s+84}{(s^2-4s+13)(s-1)}$$

$$\begin{aligned}
&= \frac{7s^2-41s+84}{((s-2)^2+3^2)(s-1)} \\
&= \frac{A}{(s-1)} + \frac{B(s-2)+3C}{(s-2)^2+3^2}
\end{aligned}$$

Putting everything on the same denominator, we obtain,

$$\begin{aligned}
A((s-2)^2+3^2)+(B(s-2)+3C)(s-1) &= 7s^2-41s+84 \\
A(s^2-4s+13)+B(s^2-3s+2)+3C(s-1) &= 7s^2-41s+84
\end{aligned}$$

We then obtain the following system of equations,

$$\begin{aligned}
A+B &= 7 \\
-4A-3B+3C &= -41 \\
13A+2B-3C &= 84
\end{aligned}$$

Solving this by matrix method to get:

$$\begin{aligned}
A &= 5 \\
B &= 2 \\
C &= -5
\end{aligned}$$

Finally,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{5}{s-1} + \frac{2(s-2)+3(-5)}{(s-2)^2+3^2}\right\} \\
f(t) &= 5e^t + 2e^{2t}\cos(3t) - 5e^{2t}\sin(3t) \\
f(t) &= 5e^t + e^{2t}(2\cos(3t) - 5\sin(3t))
\end{aligned}$$

c)

$$\begin{aligned}
sF(s)-F(s) &= \frac{2s+5}{(s+1)^2} \\
F(s) &= \frac{2s+5}{(s-1)(s+1)^2} \\
&= \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s-1}
\end{aligned}$$

Putting everything on the same denominator, we obtain,

$$\begin{aligned}
A(s-1)(s+1)+B(s-1)+C(s+1)^2 &= 2s+5 \\
A(s^2-1)+B(s-1)+C(s^2+2s+1) &= 2s+5
\end{aligned}$$

We then obtain the following system of equations,

$$\begin{aligned} A+C &= 0 \\ B+2C &= 2 \\ -A-B+C &= 5 \end{aligned}$$

Solving this by matrix method to get:

$$\begin{aligned} A &= \frac{-7}{4} \\ B &= \frac{-3}{2} \\ C &= \frac{7}{4} \end{aligned}$$

Finally,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{-7}{4(s+1)} + \frac{-3}{2(s+1)^2} + \frac{7}{4(s-1)}\right\} \\ f(t) &= \frac{-7}{4}e^{-t} - \frac{3}{2}te^{-t} + \frac{7}{4}e^{t} \end{aligned}$$

d)

$$\begin{aligned} F(s) &= \frac{7s^3-2s^2-3s+6}{s^3(s-2)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2} \end{aligned}$$

Putting everything on the same denominator, we obtain,

$$\begin{aligned} As^2(s-2) + Bs(s-2) + C(s-2) + Ds^3 &= 7s^3-2s^2-3s+6 \\ A(s^3-2s^2) + B(s^2-2s) + C(s-2) + Ds^3 &= 7s^3-2s^2-3s+6 \end{aligned}$$

We then obtain the following system of equations,

$$\begin{aligned} A+D &= 7 \\ -2A+B &= -2 \\ -2B+C &= -3 \\ -2C &= 6 \end{aligned}$$

Solving this by matrix method to get:

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -3 \\ D &= 6 \end{aligned}$$

Finally,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\Big\{\frac{1}{s}+\frac{-3}{s^3}+\frac{6}{s-2}\Big\} \\ f(t) &= 1-\frac{3t^2}{2}+6e^{2t} \end{aligned}$$

e) First, obtain an important property of Inverse Laplace transforms using the following property of Laplace transforms,

$$\begin{aligned} \mathcal{L}\{t^n f(t)\}(s) &= (-1)^n\frac{d^n}{ds^n}\{\mathcal{L}\{f\}s\} \\ \mathcal{L}\{(-t)^n f(t)\}(s) &= \{F^{(n)}\}(s) \\ \mathcal{L}^{-1}\{F^{(n)}(s)\}(t) &= (-t)^n f(t) \end{aligned} \tag{3}$$

we can use

$$\begin{aligned} F(s) &= ln\Big(\frac{s-4}{s-3}\Big) \\ &= ln(s-4)-ln(s-3) \end{aligned}$$

and plug its first derivative ($n=1$) in (3):

$$\begin{aligned} \mathcal{L}^{-1}\Big\{\frac{1}{s-4}-\frac{1}{s-3}\Big\}(t) &= (-t)^1 f(t) \\ e^{4t}-e^{3t} &= -tf(t) \\ f(t) &= \frac{e^{3t}-e^{4t}}{t} \end{aligned}$$

4 Problem 4 (Optional)

Use the method of Frobenius to find the first four nonzero terms in the series expansion about x=0 for a general solution of the equation $6x^3y^{'''}+11x^2y^{''}-2xy^{'}-(x-2)y=0$.

Let

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y^{'}(x) &= \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \\ y^{''}(x) &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} \\ y^{'''}(x) &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(n+r-2)x^{n+r-3} \end{aligned}$$

Plug this back into original ODE to obtain:

$$
\begin{aligned}
6x^3\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)(n+r-2)x^{n+r-3}+11x^2\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2} & \\
-2x\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}-(x-2)\sum_{n=0}^{\infty}a_nx^{n+r} &= 0 \\
\sum_{n=0}^{\infty}(6(n+r)(n+r-1)(n+r-2)+11(n+r)(n+r-1)-2(n+r)+2)a_nx^{n+r} & \\
-\sum_{n=0}^{\infty}a_nx^{n+r+1} &= 0
\end{aligned}
$$

Take out the $n=0$ term in the first series and express everything as a single series for powers of x^{n+r+1}.

$$
\begin{aligned}
(6(r-2)(r-1)r+11r(r-1)-2r+2)a_0x^r+\sum_{n=1}^{\infty}(6(n+r)(n+r-1)(n+r-2) & \\
+11(n+r)(n+r-1)-2(n+r)+2)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1} &= 0 \\
(6(r-2)(r-1)r+11r(r-1)-2r+2)a_0x^r+\sum_{n=0}^{\infty}((6(n+r+1)(n+r)(n+r-1) & \\
+11(n+r+1)(n+r)-2(n+r+1)+2)a_{n+1}-a_n)x^{n+r+1} &= 0
\end{aligned}
$$

Set the first term to zero to get the indicial equation.

$$
\begin{aligned}
6(r-2)(r-1)r+11r(r-1)-2r+2 &= 0 \\
6r^3-7r^2-r+2 &= 0 \\
(2r+1)(3r^2-5r+2) &= 0 \\
(2r+1)(r-1)(3r-2) &= 0
\end{aligned}
$$

The roots are then $r_1=1$, $r_2=-1/2$ and $r_3=-2/3$.

Taking $r = r_1$, the highest root, we get the following recurrence relation.

$$(6(n+r+1)(n+r)(n+r-1) + 11(n+r+1)(n+r) - 2(n+r+1) + 2)a_{n+1} - a_n = 0$$

$$\begin{aligned}
a_n &= (6(n+2)(n+1)n + 11(n+2)(n+1) - 2(n+2) + 2)a_{n+1} \\
&= (n+1)(6(n^2+2n) + 11(n+2) - 2)a_{n+1} \\
&= (n+1)(6n^2 + 23n + 20)a_{n+1} \\
&= (n+1)(n+4/3)(n+5/2)a_{n+1}
\end{aligned}$$

Plug values for n to get the first four nonzero terms to get

$$y(x) \simeq a_0\Big(x^2 + \frac{3}{10}x^3 + \frac{9}{490}x^4 + \frac{1}{2450}x^5\Big)$$