1 Integration on Surfaces

What follows are some comments on integration over parametrized surfaces. A good reference for surface integration is chapter 7 of "Vector Analysis" by Marsden and Tromba.

Recall that a parametrized surface is given by a one-to-one transformation $\phi : D \to \mathbb{R}^3$, where D is a domain in the plane $\mathbb{R}^2$. This amounts to being given three scalar functions, $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ of two variables, u and v, say. The transformation is then given by

$$\overrightarrow{r} = (x,y,z) = \phi(u,v) = (x(u,v), y(u,v), z(u,v)).$$

It is assumed that the vectors

$$\frac{\partial \overrightarrow{r}}{\partial u} = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) \quad \text{and} \quad \frac{\partial \overrightarrow{r}}{\partial v} = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$$

never vanish. These two vectors determine a vector normal (or perpendicular) to the surface $S = \phi(D)$, namely,

$$\overrightarrow{n} = \frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v} = \frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}.$$

The following formula explains how to integrate a scalar function f over S:

$$\iint_S f(x,y,z)dS = \iint_D f(\phi(u,v))|\vec{n}|\, dudv,$$

where

$$|\vec{n}| = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}.$$

A parametrized surface S always carries an orientation inherited from the orientation of the domain D in the (u,v)-plane of the transformation $\phi(u,v)$. The orientation of D is given as counterclockwise, i.e., with u considered as the "first" variable. The vector $\overrightarrow{n}$ is normal to the surface and $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \overrightarrow{n}$ is a right-handed system. If we exchange the order of the variables and put v "first" — corresponding to a clockwise orientation — then the resulting normal is

$$\frac{\partial \overrightarrow{r}}{\partial v} \times \frac{\partial \overrightarrow{r}}{\partial u} = -\frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v} = -\overrightarrow{n}$$

which points in the opposite direction from the previous normal vector and so gives S the opposite orientation. As a result, when a vector field $\overrightarrow{F} = (P,Q,R)$ is given on S and a parametrization ϕ of S is specified with an orientation on its domain D, the flux of $\vec{F}$ across S in the direction of the normal vector $\overrightarrow{n}$ is determined and is given by

$$\begin{aligned}
\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} &= \iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS \\
&= \iint_D \left\{ \overrightarrow{F}(\phi(u,v)) \cdot (\frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}|}) \right\} \left| \frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v} \right| dudv \\
&= \iint_D \overrightarrow{F}(\phi(u,v)) \cdot \frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v}\, dudv \\
&= \iint_D \left(P\frac{\partial(y,z)}{\partial(u,v)} + Q\frac{\partial(z,x)}{\partial(u,v)} + R\frac{\partial(x,y)}{\partial(u,v)} \right) dudv \\
&= \iint_D P\frac{\partial(y,z)}{\partial(u,v)}\, dudv + Q\frac{\partial(z,x)}{\partial(u,v)}\, dudv + R\frac{\partial(x,y)}{\partial(u,v)}\, dudv \\
&= \iint_S P\, dydz + Q\, dzdx + R\, dxdy,
\end{aligned}$$

which can be taken as the definition of the last integral.

A common parametrization, which will be called a **standard parametrization**, is used to describe the graph of a scalar function h of two variables. If $z = h(x, y)$ is defined on a domain D in the plane $\mathbb{R}^2$, then one lets $u = x, v = y$ and defines $\phi(x, y) = (x, y, h(x, y))$ for $(x, y) \in D$. It follows that

$$\frac{\partial \overrightarrow{r}}{\partial x} = (1, 0, \frac{\partial z}{\partial x}), \quad \frac{\partial \overrightarrow{r}}{\partial y} = (0, 1, \frac{\partial z}{\partial y})$$

so that

$$\overrightarrow{n} = (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1) \quad \text{and} \quad |\overrightarrow{n}| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$

As a result, one has the following formula for integrating a function $f(x, y, z)$ over the surface S that is the graph of h:

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, h(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dxdy.$$

Remark. When using a standard parametrization it is extremely important to realize the significance of the fact that the z-component of the vector $\overrightarrow{n}$ is 1 and hence is positive. This means, for example, if $h(x, y) \geq 0$ and one considers the solid V defined by $V = \{(x, y, z) \mid 0 \leq z \leq h(x, y)\}$, that the vector $\overrightarrow{n}$ at the point $\phi(x, y, z) = (x, y, h(x, y))$ on the "roof" of the solid V, points out of the solid. If, on the other hand, $h(x, y) \leq 0$ and one considers the solid bounded by the z-plane and the graph of h, namely $V = \{(x, y, z) \mid h(x, y) \leq z \leq 0\}$, then the vector $\overrightarrow{n}$ at the point $\phi(x, y, z) = (x, y, h(x, y)$ on the "bottom" of the solid V, points into the solid. Thinking of its significance, without reference to any solid bounded in part by the graph of h, it follows that $\overrightarrow{i}, \overrightarrow{j}$, and $\overrightarrow{n}$ form a right handed system.

In Adams (see pp. 405–406) surface integrals are discussed by projecting the surface onto one of the three coordinate planes. The case of the standard parametrization corresponds to projecting the surface onto the (x, y)-plane. If the surface S is a part of the level surface $F(x, y, z) = 0$ and it can be projected in a one-to-one way onto the (x, y)-plane this amounts to saying that for the points of S, the equation $F(x, y, z) = 0$ defines z as a function $h(x, y)$ of x and y. Implicit differentiation gives

$$F_x + F_z \frac{\partial f}{\partial x} = 0 \text{ and } F_y + F_z \frac{\partial f}{\partial z} = 0.$$

The normal vector given by the standard parametrization is $(-z_x, -z_y, 1)$ is a multiple of $\nabla F = (F_x, F_y, F_z)$ which is also normal. The multiple is $\frac{1}{F_z}$, i.e., $(-z_x, -z_y, 1) = \frac{1}{F_z}(F_x, F_y, F_z)$. Hence,

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dxdy = \frac{|\nabla F(x, y, f(x, y))|}{|F_z(x, y, f(x, y))|}\, dxdy.$$

This formula is in Adams on p. 406. Simlar formulas are in Thomas and Finney (p. 110 of the eight edition).

Remark. There are two other possibilities: projecting onto the (y, z)-plane and projecting onto the (x, z)-plane. They correspond respectively to surfaces that are the graphs of functions $k(y, z)$ and $\ell(x, z)$. Note that when using the standard parametrization $\phi(y, z) = (k(y, z), y, z)$ the normal is $(1, -x_y, -x_z)$. However, when using the standard parametrization $\phi(x, z) = (x, \ell(x, z), z)$ the normal is $(y_x, -1, y_z)$.

If S is the surface obtained by revolving the graph of $z = h(x),\ 0 \leq a \leq x \leq b$ about the z-axis, it has the equation $z = h(\sqrt{x^2 + y^2})$ with domain the annulus $a \leq \sqrt{x^2 + y^2} \leq b$. The element of area for S in the standard parametrization is

$$dS = \sqrt{1 + (f'(\sqrt{x^2 + y^2}))^2}\, dxdy$$

and $\overrightarrow{n} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = (1, 0, \frac{xf'(\sqrt{x^2+y^2}\ )}{\sqrt{x^2+y^2}}) \times (0, 1, \frac{yf'(\sqrt{x^2+y^2}\ )}{\sqrt{x^2+y^2}}) = (\frac{-xf'(\sqrt{x^2+y^2}\ )}{\sqrt{x^2+y^2}}, \frac{-yf'(\sqrt{x^2+y^2}\ )}{\sqrt{x^2+y^2}}, 1)$

If we use the parametrization

$$x = r\cos\theta,\ \ y = r\sin\theta, z = h(r), \quad 0 \le \theta \le 2\pi,\ \ a \le r \le b,$$

we have $dS = r\sqrt{1 + f'(r)^2}\, dr d\theta$ and

$$\overrightarrow{n} = \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = (\cos\theta, \sin\theta, h'(r)) \times (-r\sin\theta, r\cos\theta, 0) = (-rh'(r)\cos\theta, -rh'(r)\sin\theta, r).$$

If S is the surface obtained by revolving the graph of $z = h(x) \ge 0,\ \le a \le x \le b$ about the zx-axis, it has the equation $\sqrt{y^2 + z^2} = f(x)$ and the parametrization

$$\overrightarrow{r} = (x, f(x)\cos\theta, f(x)\sin\theta),\ \ 0 \le \theta \le 2\pi,\ \ a \le x \le b$$

from which one gets

$$\begin{aligned} \overrightarrow{n} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta} &= (1, f'(x)\cos\theta, f'(x)\sin\theta) \times (0, -f(x)\cos\theta, -f(x)\sin\theta) \\ &= (f'(x)f(x), -f(x)\cos\theta, -f(x)\sin\theta) \end{aligned}$$

$$dS = f(x)\sqrt{1 + f'(x)^2}\, dx d\theta.$$