1 The Implicit Function Theorem

Suppose that \((a, b)\) is a point on the curve \(F(x, y) = 0\) where and suppose that this equation can be solved for \(y\) as a function of \(x\) for all \((x, y)\) sufficiently near \((a, b)\). Then this part of the curve is the graph of a function \(y = \varphi(x)\) on some interval \(|x - a| < h\) with \(\varphi(a) = b\). If \(\varphi'(x)\) exists, we can compute it by differentiating both sides of the equation \(F(x, \varphi(x)) = 0\) with respect to \(x\) to get

\[
\frac{\partial F}{\partial x}(x, \varphi(x)) + \frac{\partial F}{\partial y}(x, \varphi(x))\varphi'(x) = 0
\]

providing that the partial derivatives exist. If \(\frac{\partial F}{\partial y}(x, \varphi(x)) \neq 0\), we can solve for \(\varphi'(x)\) and obtain the well known formula

\[
\varphi'(x) = -\frac{\frac{\partial F}{\partial x}(x, \varphi(x))}{\frac{\partial F}{\partial y}(x, \varphi(x))}
\]

or, more classically,

\[
\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.
\]

The precise conditions under which the existence of \(h\) and \(\varphi\) is assured are furnished by the following theorem, which is the Implicit Function Theorem for functions of two variables.

**Theorem 1.** If \(F(a, b) = 0\) and \(F(x, y)\) is continuously differentiable on some open disk with center \((a, b)\) then, if \(\frac{\partial F}{\partial y}(a, b) \neq 0\), there exists an \(h > 0\) and a unique function \(\varphi(x)\) defined for \(|x - a| < h\) such that \(\varphi(a) = b\) and \(F(x, \varphi(x)) = 0\) for \(|x - a| < h\). Moreover, on \(|x - a| < h\), the function \(\varphi(x)\) is continuously differentiable and

\[
\varphi'(x) = -\frac{\frac{\partial F}{\partial x}(x, \varphi(x))}{\frac{\partial F}{\partial y}(x, \varphi(x))}
\]

There is a corresponding theorem for the case where \(\frac{\partial F}{\partial y}(a, b) \neq 0\). In this case the curve \(F(x, y)\) is the graph of a function of \(x = \psi(y)\) near the point \((a, b)\).

**Example.** Except for the two points \((\pm 1, 0)\), the curve \(x^2 + y^2 = 1\) consists of the two continuously differentiable functions \(y = \pm \sqrt{1 - x^2}, -1 < x < 1\). Notice that for \(F(x, y) = x^2 + y^2\) we have \(\frac{\partial F}{\partial y} = 2y\) which is zero when \(y = 0\). The points \((\pm 1, 0)\) lie on the two branches \(x = \pm \sqrt{1 - y^2}, -1 < y < 1\).

The general Implicit Function Theorem gives condition under which a system of equations

\[
F_1(x_1, \ldots , x_m, y_1, \ldots , y_n) = 0 \\
F_2(x_1, \ldots , x_m, y_1, \ldots , y_n) = 0 \\
\vdots \\
F_n(x_1, \ldots , x_m, y_1, \ldots , y_n) = 0
\]

can be solved for \(y_1, \ldots , y_n\) as functions of \(x_1, \ldots , x_m\), say \(y_i = \varphi_i(x_1, \ldots , x_m)\). Differentiating the equation

\[
F_i(x_1, \ldots , x_m, \varphi_1(x_1, \ldots , x_m), \ldots , \varphi_n(x_1, \ldots , x_m)) = 0
\]

with respect to \(x_j\) we get

\[
\frac{\partial F_i}{\partial x_j} = \frac{\partial F_i}{\partial y_1} \frac{\partial \varphi_1}{\partial x_j} + \ldots + \frac{\partial F_i}{\partial y_n} \frac{\partial \varphi_n}{\partial x_j}.
\]

Using the partial Jacobians

\[
D_x F = \begin{bmatrix}
\frac{\partial F_1}{\partial x_1} & \ldots & \frac{\partial F_1}{\partial x_m} \\
\frac{\partial F_2}{\partial x_1} & \ldots & \frac{\partial F_2}{\partial x_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} & \ldots & \frac{\partial F_n}{\partial x_m}
\end{bmatrix}, \quad
D_y F = \begin{bmatrix}
\frac{\partial F_1}{\partial y_1} & \ldots & \frac{\partial F_1}{\partial y_n} \\
\frac{\partial F_2}{\partial y_1} & \ldots & \frac{\partial F_2}{\partial y_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial y_1} & \ldots & \frac{\partial F_n}{\partial y_n}
\end{bmatrix}.
\]
These equations with $1 \leq i \leq n$ can be written in the matrix form

$$D_x F + D_y F D \varphi = 0.$$ 

If $D_y F$ is invertible, i.e., if $|D_y F| \neq 0$, we get

$$D \varphi = -D_y F^{-1} D_x F.$$ 

Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$ and let

$$F(x, y) = (F_1(x, y), \ldots, F_n(x, y)).$$

**Theorem 2 (Implicit Function Theorem).** If $F(a, b) = 0$ and $F(x, y)$ is continuously differentiable on some open disk with center $(a, b)$ then, if $|D_y F(a, b)| \neq 0$, there exists an $h > 0$ and a unique function $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ defined for $|x - a| < h$ such that $\varphi(a) = b$ and $F(x, \varphi(x)) = 0$ for $|x - a| < h$. Moreover, on $|x - a| < h$, the function $\varphi(x)$ is continuously differentiable and

$$D\varphi(x) = -D_y F(x, \varphi(x))^{-1} D_x F(x, \varphi(x)).$$

**Example.** Consider the equation $F(x, y) = 0$ where

$$F_1(x, y) = x_1^2 + 2x_2 + y_1^2 + 2y_2 - 8 = 0,$$

$$F_2(x, y) = x_1 - x_2^2 + y_1 - y_2^2 + 3 = 0.$$ 

If $a = (1, 1)$, $b = (1, 2)$, we have $F(a, b) = 0$ and

$$|D_y F(a, b)| = \begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} = -10 \neq 0.$$ 

By the Implicit Function Theorem one has, for $x = (x_1, x_2)$ sufficiently close to $(1, 1)$,

$$y = (y_1, y_2) = (\varphi_1(x), \varphi_2(x)) = \varphi(x)$$

with $\varphi(1, 1) = (1, 2)$ and

$$D\varphi(x) = - \begin{bmatrix} 2\varphi_1(x) & 2 \\ 1 & -2\varphi_2(x) \end{bmatrix}^{-1} \begin{bmatrix} 2x_1 & 2 \\ 1 & -2\varphi_2(x) \end{bmatrix}.$$ 

An immediate consequence of the Implicit Function Theorem is the following theorem, known as the Inverse Function Theorem.

**Theorem 3 (Inverse Function Theorem).** Let $y = f(x)$, where $y = (y_1, y_2, \ldots, y_n)$ and $x = (x_1, x_2, \ldots, x_n)$. If $DF(a)$ is invertible, then, for $y$ near $b = f(a)$ and $x$ near $a$, we have $x = g(y)$ and $Dg(b) = Df(a)^{-1}$. 