1 The Continuity Equation

Imagine a fluid flowing in a region R of the plane in a time dependent fashion. At each point $(x, y) \in \mathbb{R}^2$ it has a velocity $\overrightarrow{v} = \overrightarrow{v}(x, y, t)$ at time t. Let $\rho = \rho(x, y, t)$ be the density of the fluid at (x, y) at time t. Let P be any point in the interior of R and let D_r be the closed disk of radius $r > 0$ and center P. The mass of fluid inside D_r at any time t is

$$\iint_{D_r} \rho \, dxdy.$$

If matter is neither created nor destroyed inside D_r, the rate of decrease of this quantity is equal to the flux of the vector field $\overrightarrow{J} = \rho\overrightarrow{v}$ across C_r, the positively oriented boundary of D_r. We therefore have

$$\frac{d}{dt}\iint_{D_r} \rho \, dxdy = -\int_{C_r} \overrightarrow{J} \cdot \overrightarrow{N} \, ds,$$

where $\vec{N}$ is the outer normal and ds is the element of arc length. Notice that the minus sign is needed since positive flux at time t represents loss of total mass at that time. Also observe that the amount of fluid transported across a small piece ds of the boundary of D_r at time t is $\rho\overrightarrow{v} \cdot \overrightarrow{N} \, ds$. Differentiating under the integral sign on the left-hand side and using the flux form of Green's Theorem on the right-hand side, we get

$$\iint_{D_r} \frac{\partial \rho}{\partial t} \, dxdy = -\iint_{D_r} \nabla \cdot \overrightarrow{J} \, dxdy.$$

Gathering terms on the left-hand side, we get

$$\iint_{D_r} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \overrightarrow{J}\right) dxdy = 0.$$

If the integrand was not zero at P it would be different from zero on D_r for some sufficiently small r and hence the integral would not be zero which is not the case. Hence the integrand is zero at P and, since P was arbitrary, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \overrightarrow{J} = 0.$$

Conversely, if this equation holds then matter is neither created nor destroyed in R. For this reason this equation is called the **conservation equation**. It is also known as the **continuity equation**. Since

$$\nabla(\rho\overrightarrow{v}) = \nabla\rho \cdot \overrightarrow{v} + \rho\nabla \cdot \overrightarrow{v}$$

the continuity equation can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla\rho \cdot \overrightarrow{v} + \rho\nabla \cdot \overrightarrow{v} = 0.$$

It follows that the vector field J is incompressible at any time t if and only if $\frac{\partial \rho}{\partial t} = 0$. The vector field $\overrightarrow{v}$ is incompressible at any time t if and only if

$$\frac{\partial \rho}{\partial t} + \nabla\rho \cdot \overrightarrow{v} = 0.$$

If ρ does not depend on x, y then $\overrightarrow{v}$ is incompressible if and only if $\overrightarrow{J}$ is.

2 The Heat Equation

Imagine a metal plate in the shape of some region R of the plane. Let $T = T(x, y, t)$ denote the temperature of the plate at the point (x, y) at the time t. In the theory of heat flow, one assumes that heat flows from hot to cold regions. As a result, the heat flow is the time dependent vector field $\vec{v} = -\nabla T$, where the gradient is taken relative to the space variables x and y. If c denotes the specific heat and ρ is the mass density, then the quantity of heat inside a disk D in the interior of R is at time t

$$\iint_D \rho c T \, dxdy.$$

If there are no heat sources or sinks in D then the rate at which this quantity increases is equal the rate at heat is gained at the boundary C of D. Since heat flows in the direction of $-\nabla T$ and the rate of flow is equal to $\kappa|\nabla T|$, where κ is the conductivity of the material, heat is gained at the boundary of D at the rate

$$\int_C \kappa \nabla T \cdot \vec{N} \, ds = \iint_D \nabla \cdot \nabla T \, dxdy.$$

It follows that

$$\frac{d}{dt} \iint_D \rho c T \, dxdy = \iint_D \nabla \cdot \nabla T \, dxdy$$

and hence that

$$\iint_D \rho c \frac{\partial T}{\partial t} \, dxdy = \iint_D \nabla \cdot \nabla T \, dxdy$$

Since D is arbitrary, it follows that

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot \nabla T.$$

This equation can be written in the form

$$\frac{\partial T}{\partial t} = k \nabla \cdot \nabla T,$$

where $k = \kappa / c\rho$ is called the **diffusivity**. This equation is known as the **heat equation**. Since

$$\nabla \cdot \nabla T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},$$

the heat equation can be written in the form

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

or, equivalently, in the form $\frac{\partial T}{\partial t} = k \Delta T$, where by definition

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

the **Laplacian operator**.

From the above, it follows that if one has a distribution of heat on a metal plate R and the heat distribution T does not change with time — the so-called **steady state** — then

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \text{ on } R.$$

This equation is known as **Laplace's equation**. The solutions of Laplace's equation are known as **harmonic functions**. The functions

$$xy, \quad x^2 - y^2, \quad e^x \cos y, \quad e^x \sin y.$$

are examples of harmonic functions on $\mathbb{R}^2$. The function $\log(x^2 + y^2)$ is harmonic on $\mathbb{R}^2 - \{(0,0)\}$. The gradient of this function is the vector field

$$\vec{F} = \frac{x}{x^2 + y^2} \vec{i} + \frac{y}{x^2 + y^2} \vec{j}.$$

This vector field is the field produced by a uniformly distributed electrical charge of unit charge density along the z-axis. Indeed, an easy calculation shows that

$$\overrightarrow{F} = \int_{-\infty}^{\infty} \frac{x\vec{i} + y\vec{j} - z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}\, dz.$$