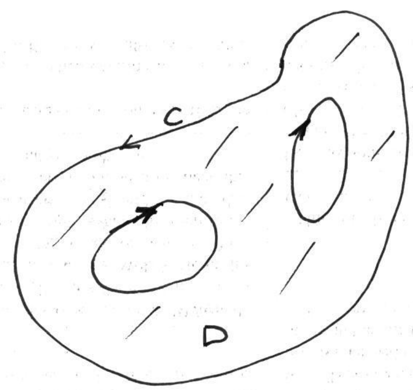
1 Green's Theorem

Green's theorem states that a line integral around the boundary of a plane region D can be computed as a double integral over D. More precisely, if D is a "nice" region in the plane and C is the boundary of D with C oriented so that D is always on the left-hand side as one goes around C (this is the positive orientation of C), then

$$\int_C P dx + Q dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy$$

if the partial derivatives of P and Q are continuous on D.

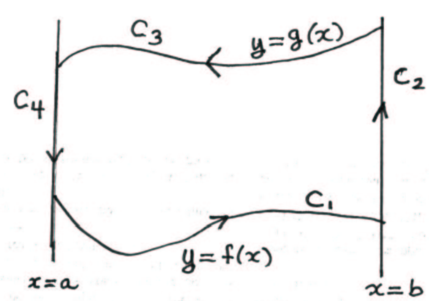


Regions that are simultaneously of type I and II are "nice" regions, i.e., Green's theorem is true for such regions. The next two propositions prove this.

Theorem 1. *If D is a region of type I then*

$$\int_C P dx = -\iint_D \frac{\partial P}{\partial y}\, dxdy.$$

Proof. If $D = \{(x, y) \mid a \le x \le b,\ f(x) \le y \le g(x)\}$ with $f(x)$, $g(x)$ continuous on $a \le x \le b$, we have $C = C_1 + C_2 + C_3 + C_4$, where C_1, C_2, C_3, C_4 are as shown below.



Since the region is of type I, we have

$$\iint_D \frac{\partial P}{\partial y}\, dxdy = \int_a^b \left[P(x, g(x)) - P(x, f(x))\right] dx.$$

Using the standard parametrizations of C_1 and C_3, we have

$$\int_a^b P(x, f(x))\, dx = \int_{C_1} P\, dx, \quad \int_a^b P(x, g(x))\, dx = -\int_{C_3} P\, dx$$

We thus obtain

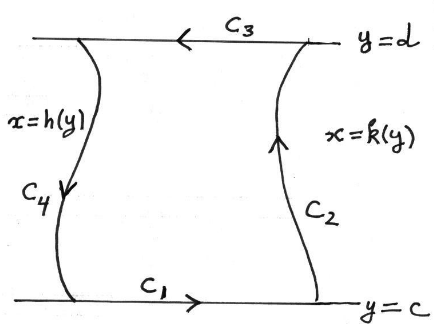
$$\iint_D \frac{\partial P}{\partial y}\, dxdy = \int_{C_1} P\, dx + \int_{C_3} P\, dx = \int_C P\, dx$$

since the line integral of $P\, dx$ is zero on C_2 and C_4 as x is constant there. □

Theorem 2. *If D is a region of type II then*

$$\int_C Q\, dy = \iint_D \frac{\partial Q}{\partial x}\, dxdy.$$

Proof. If $D = \{(x, y) \mid h(y) \leq x \leq k(y),\ c \leq y \leq d\}$ with $h(y)$, $k(y)$ continuous on $c \leq y \leq d$, we have $C = C_1 + C_2 + C_3 + C_4$, where C_1, C_2, C_3, C_4 are as shown below.



Since the region is of type II, we have

$$\iint_D \frac{\partial Q}{\partial y}\, dxdy = \int_c^d \big[Q(k(y), y) - Q(h(y), y)\big]\, dy.$$

Using the standard parametrizations of C_2 and C_4, we have

$$\int_c^d Q(k(y), y)\, dy = \int_{C_2} Q\, dy, \quad \int_c^d Q(h(y), y)\, dx = -\int_{C_4} Q\, dy$$

We thus obtain

$$\iint_D \frac{\partial Q}{\partial x}\, dxdy = \int_{C_2} Q\, dy + \int_{C_4} Q\, dy = \int_C Q\, dy$$

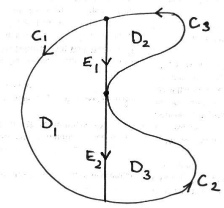
since the line integral of $Q\, dx$ is zero on C_1 and C_3 as y is constant there. □

Putting these two theorems together, we obtain

Theorem 3. *If D is a region of the plane that is simultaneously type I and type II then*

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Green' Theorem can easily be extended to any region that can be decomposed into a finite number of regions with are both type I and type II. Such regions we call "nice". Fortunately, most regions are nice. For example, consider the region below.



Since D is the union of D_1,D_2 and D_3, we have

$$\iint_D = \iint_{D_1} + \iint_{D_2} + \iint_{D_3}.$$

Since the regions D_1, D_2, D_3 are all type I and type II and the positively oriented boundaries of D_1, D_2 D_3 are respectively $C_1 - E_2 - E_1$, $E_1 + C_2$, $E_2 + C_2$, we have

$$\iint_{D_1} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy = \int_{C_1} P\,dx + Q\,dy - \int_{E_1} P\,dx + Q\,dy - \int_{E_2} P\,dx + Q\,dy,$$

$$\iint_{D_2} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy = \int_{E_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy,$$

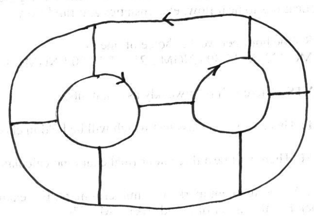
$$\iint_{D_3} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy = \int_{E_2} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy.$$

Adding these equations, we get

$$\iint_{D} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy = \int_1 CP\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy,$$

where $C = C_1 + C_2 + C_3$ is the positively oriented boundary of D. This yields Green's Theorem for D.

The reader is invited to prove Green's Theorem for the region below using the given decomposition into regions which are type I and type II.



2 The flux form of Green's Theorem

Let R be a region for which Green's Theorem holds and let C be the positively oriented boundary of R. For each point P on C let $\overrightarrow{T}$ be the unit tangent vector at P and let $\overrightarrow{N} = \overrightarrow{T} \times \vec{k}$, where $\vec{k}$ is the unit normal to the x, y-plane.

Theorem 4. *If $\overrightarrow{F} = P\vec{i} + Q\vec{j}$ is a twice continuously differentiable vector field on R, then*

$$\int_C \overrightarrow{F} \cdot \overrightarrow{N}\, ds = \iint_R (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})\, dxdy.$$

Proof. Since $\overrightarrow{u} \cdot (\overrightarrow{v} \times \vec{w} = (\overrightarrow{w} \times \overrightarrow{u}) \cdot \overrightarrow{v}$, we have

$$\int_C \overrightarrow{F} \cdot (\overrightarrow{T} \times \vec{k})\, ds = \int_C (\vec{k} \times \overrightarrow{F}) \cdot \overrightarrow{T}\, ds = \int_C -Q\, dx + P\, dy = \iint_R (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})\, dxdy.$$

□

This theorem is called the flux form of Green's Theorem since

$$\int_C \overrightarrow{F} \cdot \overrightarrow{N}\, ds$$

is the flux of $\overrightarrow{F}$ across C. The function $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is called the divergence of the vector field $\overrightarrow{F} = P\vec{i} + Q\vec{j}$ and is denoted by $\mathrm{div}(\overrightarrow{F})$. For this reason, Theorem 4 is also called the 2-dimensional Divergence Theorem. Note that,if $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$, we have

$$\mathrm{div}(\overrightarrow{F}) = \nabla \cdot \overrightarrow{F}.$$

The vector field $\nabla \times \overrightarrow{F} = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}$ is the called the curl of the vector field $\overrightarrow{F}$ and is also denoted by $\mathrm{curl}(\overrightarrow{F})$. The first form of Green's Theorem can be stated as

$$\int_C \overrightarrow{F} \cdot \overrightarrow{T} ds = \iint_R \mathrm{curl}(\overrightarrow{F}) \cdot \vec{k}\, dxdy.$$

These two equivalent forms of Green's Theorem in the plane give rise to two distinct theorems in three dimensions. The usual form of Green's Theorem corresponds to Stokes' Theorem and the flux form of Green's Theorem to Gauss' Theorem, also called the Divergence Theorem. In Adams' textbook, in Chapter 9 of the third edition, he first derives the Gauss theorem in §9.3, followed, in Example 6 of §9.3, by the two dimensional version of it that has here been referred to as the flux form of Green's Theorem. He then uses this two dimensional version in §9.4 to derive the usual form of Green's Theorem which he uses to prove Stokes' Theorem, the three dimensional version of Green's Theorem. [In this COURSE PAK (see **Stokes' theorem**) it is also shown how to deduce Stokes' Theorem from Green's Theorem.]

Green's Theorem can be used to give a physical interpretation of the curl in the case $\overrightarrow{F}$ is the velocity field $\overrightarrow{v}$ of a flow. If C_r is a circle of radius r with center P, then the average value of the angular velocity $\omega_r = \overrightarrow{v} \cdot \overrightarrow{T}/r$ on C_r is

$$\overline{\omega}_r = \frac{1}{2\pi r^2} \int_{C_r} \overrightarrow{v} \cdot \overrightarrow{T}\, ds.$$

If D_r is the closed disk with boundary C_r, the average value of $\mathrm{curl}(\overrightarrow{v}) \cdot \vec{k}$ on D_r is

$$\frac{1}{\pi r^2} \iint_{D_r} \mathrm{curl}(\overrightarrow{v}) \cdot \vec{k}\, dxdy = \frac{2\overline{v}_T}{r} = 2\overline{\omega_r}.$$

Taking the limit as $r \to 0$, we find that that the angular velocity of the flow around P is

$$\omega = \lim_{r \to 0} \omega_r = \frac{1}{2}\mathrm{curl}(\overrightarrow{v})(P) \cdot \vec{k}$$

and hence that $\mathrm{curl}(\overrightarrow{v})(P) = 2\omega\vec{k}$. For this reason, we sometimes denote $\mathrm{curl}(\vec{v})$ by $\mathrm{rot}(\overrightarrow{v})$. The vector field $\vec{v}$ is said to be **irrotational** if $\mathrm{curl}(\overrightarrow{v}) = 0$.

Using the flux form of Green's Theorem we can, in the same way, give a physical interpretation of $\mathrm{div}(\overrightarrow{v})(P)$. The flux of $\overrightarrow{v}$ across C_r per unit area of D_r is

$$\frac{1}{\pi r^2} \int_{C_r} \overrightarrow{v} \cdot \overrightarrow{T}\, ds = \frac{1}{\pi r^2} \iint_{D_r} \mathrm{div}(\overrightarrow{v})\, dxdy = \mathrm{div}(\overrightarrow{v})(Q_r)$$

for some point Q_r in D_r. Taking the limit as $r \to 0$, we find that $\mathrm{div}(\overrightarrow{v})(P)$ measures the rate of change of the quantity of fluid or gas flowing from P per unit area. For this reason, P since called a **source** if $\mathrm{div}(\overrightarrow{v})(P) > 0$ and a **sink** if $\mathrm{div}(\overrightarrow{v})(P) < 0$. The vector field $\overrightarrow{v}$ is said to be **incompressible** if $\mathrm{div}(\overrightarrow{v}) = 0$.