The fundamental theorem of integral calculus states that

$$\int_a^b f'(x)dx = f(b) - f(a)$$

for any continuously differentiable function $f(x)$ on the interval $a \le x \le b$ or, equivalently, that

$$\int_a^b df = f(b) - f(a),$$

where $df = f'(x)dx$. The corresponding result for line integrals is the following

Theorem 1. *Let $f(x, y)$ be differentiable on the curve C which has a parametric representation $\overrightarrow{r} = \overrightarrow{r}(t) = (x(t), y(t), z(t))$, $a \le t \le b$, with $x(t), y(t), z(t)$ continuously differentiable for $a \le t \le b$. Then, if $A = (x(a), y(a), z(a)$ and $B = (x(b), y(b), z(b)$, we have*

$$\int_C \nabla f \cdot d\overrightarrow{r} = f(B) - f(A).$$

Proof. Computing the line integral, we get

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}\, dt.$$

Setting $\phi(t) = f(x(t), y(t), z(t))$ and using the chin rule, we have

$$\phi'(t) = \nabla f(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}.$$

It follows that

$$\int_C \nabla f \cdot d\overrightarrow{r} = \int_a^b \phi'(t)\, dt = \phi(b) - \phi(a) = f(B) - f(A).$$

□

Since $\nabla f \cdot d\vec{r} = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = df$, Theorem 1 can also be stated as

$$\int_C \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = f(B) - f(A)$$

or simply as $\int_C df = f(B) - f(A)$.

It follows from Theorem 1 that

$$\int_{C_1} \nabla f \cdot d\overrightarrow{r} = \int_{C_2} \nabla f \cdot d\overrightarrow{r}$$

for any two smooth paths from A to B, i.e., the integral is **path independent**. A vector field $\overrightarrow{F} = (P, Q, R)$ is is to be **conservative** if the line integral

$$\int_C \nabla f \cdot d\overrightarrow{r} = \int_C P\, dx + Q\, dy + R\, dz$$

is independent of the smooth path joining any two fixed point or, equivalently that the integral be zero for any smooth closed curve C. This is true if $\overrightarrow{F} = \nabla\phi$. In this case $\overrightarrow{F}$ is called a **gradient field** with **potential function** ϕ. The converse is also true.

Theorem 2. *A conservative vector field $\overrightarrow{F}$ is a gradient field.*

Proof. We fix a point A and let C to the point B with coordinates (x, y, z). If $\overrightarrow{F} = (P, Q, R)$ then define the $\phi(x, y, z)$ by

$$\phi(x, y, z) = \int_A^B P\,dx + Q\,dy + R\,dz.$$

If E is the line segment from (x, y, z) to $(x + h, y, z)$, we have

$$\phi(x + h, y, z) - \phi(x, y, z) = \int_E P\,dx + Q\,dy + R\,dz = \int_0^1 P(x + th, y, z)\,dx = hP(x + t_1 h, y, z),$$

where $0 \leq t_1 \leq 1$. It follows that

$$\frac{\partial \phi}{\partial x} = \lim_{h \to 0} \frac{\phi(x + h, y, z) - \phi(x, y, z)}{h} = \lim_{h \to 0} P(x + t_1 h, y, z) = P(x, y, z).$$

Similarly, $\frac{\partial \phi}{\partial y} = Q$ and $\frac{\partial \phi}{\partial y} = R$ so that $\overrightarrow{F} = \nabla \phi$. □

A necessary condition for $F = P\vec{i} + Q\vec{j} + R\vec{k}$ to be conservative is that $\mathrm{curl}(\overrightarrow{F})$ be zero, where

$$\mathrm{curl}(\overrightarrow{F}) = \nabla \times \overrightarrow{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}.$$

The converse is not true. For example, if

$$\overrightarrow{F} = \frac{-y}{x^2 + y^2}\vec{i} + \frac{x}{x^2 + y^2}\vec{j}$$

then the curl of $\overrightarrow{F}$ is zero while, if C is the circle $x = \cos(t)$, $y = \sin(t)$, $z = 0$, $0 \leq t \leq 2\pi$, we have

$$\int_C \overrightarrow{F} \cdot d\vec{r} = \int_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = 2\pi.$$