This is a discussion of some easy special cases of the inhomogeneous wave equation with Dirichlet and Neumann boundary conditions.

## 1. Dirichlet BC

Recall from Wave equation handout, Part 2 that we were solving the IBVP

$$u_{tt} = c^2 u_{xx} + H(x,t), \quad u(x,0) = f(x), u_t(x,0) = g(x) \quad u(0,t) = 0 = u(L,t). \tag{1}$$

To find the solution, we expanded the inhomogeneous term $H(x,t)$ into Fourier series

$$H(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{\pi n x}{L}\right), \tag{2}$$

and looked for solutions of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{\pi n x}{L}\right).$$

Below we consider a simple special case, when an infinite expansion (3) consists of a *single term*, say

$$H(x,t) = h(t) \sin\left(\frac{\pi m x}{L}\right), \tag{3}$$

for some $m > 0$. We would like to find a particular solution of the form

$$u_p(x,t) = u(t) \sin\left(\frac{\pi m x}{L}\right),$$

where $u(t)$ is an unknown function.

The function $u(t)$ satisfies the differential equation

$$u''(t) + \left(\frac{\pi m c}{L}\right)^2 u(t) = h(t). \tag{4}$$

The system of fundamental solutions is $\{\sin\left(\frac{\pi mct}{L}\right), \cos\left(\frac{\pi mct}{L}\right)\}$.

We shall now consider a very special case when the function $h(t)$ has the form

$$h(t) = e^{\alpha t}(P_n(t)\cos(\beta t) + Q_n(t)\sin(\beta t)), \tag{5}$$

where $P_n, Q_n$ are polynomials of degree $\leq n$, and $\max(\deg P, \deg Q) = n$.

Such equations can be solved by method of undetermined coefficients. We consider two cases:

i) $\alpha \neq 0$, or $\alpha = 0$ and $\beta \neq \pi mc/L$. In that case there exists a solution of (5) of the form

$$e^{\alpha t}(p_n(t)\cos(\beta t) + q_n(t)\sin(\beta t)),$$

where $p_n, q_n$ are also polynomials of degree $\leq n$, different from $P_n$ and $Q_n$.

ii) $\alpha = 0$ and $\beta = \pi mc/L$. In that case there exists a solution of (5) of the form

$$t(p_n(t)\cos(mt) + q_n(t)\sin(mt)),$$

where $p_n, q_n$ are also polynomials of degree $\leq n$, different from $P_n$ and $Q_n$.

After we find a solution $u_p$, a general solution has the form $u = u_p + v$, where $v$ satisfies

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x) - u_p(x,0), u_t(x,0) = g(x) - (u_p)_t(x,0) \quad u(0,t) = 0 = u(L,t).$$

It can be solved by methods discussed in wave equation handout number 1.

1.1. **Example 1.** . Solve

$$u_{tt} = 4u_{xx} + \sin(4x)e^t(1+2t), \quad u(0,t) = 0 = u(\pi,t).$$

**Solution:** Here $L = \pi$, $c = 2$ and $m = 4$. We look for solutions in the form $u_p(x,t) = \sin(4x)u(t)$. We have $\pi mc/L = \pi \cdot 4 \cdot 2/\pi = 8$.

The function $u(t)$ satisfies

$$u''(t) = -64u(t) + e^t(1+2t) \tag{6}$$

The function $h(t) = e^t(1+2t)$, so $\alpha = 1 \neq 0, \beta = 0$, and $n = 1$.

We try to find solutions by method of undetermined coefficients. Since $\alpha \neq 0$, we are in case (i), hence we look for solutions of the form

$$u(t) = e^t(a + b \cdot t),$$

where $a$ and $b$ are the undetermined coefficients that we have to find. Substituting into (6), we find that

$$[e^t(a + b \cdot t)]'' = e^t[a + 2b + bt] = -64u(t) + e^t(1+2t) = e^t[-64a + 1 + (-64b+2)t].$$

Equating the coefficients in the previous formula, we see that $a + 2b = -64a + 1, b = -64b + 2$. We first solve for $b$ and find that $b = 2/65$. Finally, from the first equation we see that $65a = 61/65$, and so $a = 61/(65)^2$. So, the solution to (6) that we found is

$$u_p(t) = e^t\left(\frac{61}{65^2} + \frac{2t}{65}\right)$$

1.2. **Example 2.** . Solve

$$u_{tt} = u_{xx} + \sin(2x) \cdot 2t\cos(2t), \quad u(0,t) = 0 = u(\pi, t).$$

**Solution:** Here $L = \pi$, $c = 1$ and $m = 2$. We look for solutions in the form $u_p(x,t) = \sin(2x)u(t)$. We have $\pi mc/L = \pi \cdot 2 \cdot 2/\pi = 2$.

The function $u(t)$ satisfies

$$u''(t) = -4u(t) + 2t\cos(2t) \tag{7}$$

The function $h(t) = 2t\cos(2t)$, so $n = 1$, $\alpha = 0$, and $\beta = 2 = \pi mc/L$, and we are in case (ii).

We try to find solutions by method of undetermined coefficients. Since we are in case (ii), we look for solutions of the form

$$u(t) = t[\cos(2t)(a + bt) + \sin(2t)(c + dt)],$$

where $a, b, c, d$ are the undetermined coefficients that we have to find. We first compute the second derivative, and find (after a long calculation!) that

$$u(t)'' = \cos(2t)[4c + 8dt + 2b - 4at - 4bt^2] + \sin(2t)[-4a - 8bt + 2d - 4ct - 4dt^2]. \tag{8}$$

On the other hand, the right-hand side of (7) is equal to

$$-4u(t) + 2t\cos(2t) = \cos(2t)[-4at - 4bt^2 + 2t] + \sin(2t)[-4ct - 4dt^2]. \tag{9}$$

Equating the coefficients of $\cos(2t)$ and $\sin(2t)$ in (8) and (9), we see that $4c+8dt+2b - 4at - 4bt^2 = -4at - 4bt^2 + 2t$, and $-4a - 8bt + 2d - 4ct - 4dt^2 = -4ct - 4dt^2$. After cancelations, we get the following system of equations:

$$\begin{cases} 8d = 2; \\ 2b + 4c = 0; \\ -8b = 0; \\ 2d - 4a = 0. \end{cases} \tag{10}$$

It follows that $b = c = 0, d = 1/4, a = 1/8$.

The solution to (7) that we found is

$$u_p(t) = t\cos(2t)/8 + t^2\sin(2t)/4.$$

## 2. Neumann BC

Recall from Wave equation handout, Part 3, that we were solving an IBVP

(11) $u_{tt} = c^2 u_{xx} + H(x,t), \quad u(x,0) = f(x), u_t(x,0) = g(x) \quad u_x(0,t) = 0 = u_x(L,t).$

We expanded

$$H(x,t) = \frac{h_0(t)}{2} + \sum_{n=1}^{\infty} h_n(t) \cos\left(\frac{\pi n x}{L}\right), \tag{12}$$

and looked for solutions of the form

$$u(x,t) = u_0(t) + \sum_{n=1}^{\infty} u_n(t) \cos\left(\frac{\pi n x}{L}\right).$$

Here we consider a special case when an infinite expansion (12) consists of a single term,

$$H(x,t) = h(t) \cos\left(\frac{\pi m x}{L}\right), \quad m > 0.$$

Analogously to the Dirichlet case, we look for particular solutions of the form

$$u_p(x,t) = u(t) \cos\left(\frac{\pi m x}{L}\right),$$

where $u(t)$ is an unknown function.

The function $u(t)$ satisfies the differential equation

$$u''(t) + \left(\frac{\pi m c}{L}\right)^2 u(t) = h(t).$$

This equation is identical to (4), and is solved exactly as in the section 1.