This is a continuation of Part 2 (Dirichlet BC).

1. Inhomogeneous equation

1.1. Neumann BC. We would like to solve IBVP

(1) $u_{tt} = c^2 u_{xx} + H(x,t), \quad u(x,0) = f(x), u_t(x,0) = g(x) \quad u_x(0,t) = 0 = u_x(L,t).$

We look for solutions of the form

$$u(x,t) = u_0(t) + \sum_{n=1}^{\infty} u_n(t) \cos\left(\frac{\pi n x}{L}\right),$$

which will automatically satisfy the Neumann boundary conditions in (1).

To find $u_n(t)$, we first expand $H(x,t)$ in *cosine* Fourier series in x, where the coefficients are functions of t:

$$H(x,t) = \frac{h_0(t)}{2} + \sum_{n=1}^{\infty} h_n(t) \cos\left(\frac{\pi n x}{L}\right),$$

where

$$h_0(t) = \frac{2}{L}\int_0^L H(x,t)dx, \quad h_n(t) = \frac{2}{L}\int_0^L H(x,t)\cos\left(\frac{\pi n x}{L}\right)dx.$$

We next expand $f(x)$ in cosine Fourier series,

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{L}\right), \quad A_0 = \frac{2}{L}\int_0^L f(x)dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right)dx.$$

We also expand $g(x)$ in cosine Fourier series,

$$g(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{\pi n x}{L}\right), \quad b_0 = \frac{2}{L}\int_0^L g(x)dx, \quad b_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{\pi n x}{L}\right)dx.$$

Let $B_0 = b_0$, and let

$$B_n = \frac{b_n L}{\pi n c} = \frac{2}{\pi n c}\int_0^L g(x)\cos\left(\frac{\pi n x}{L}\right)dx.$$

We first consider $n = 0$. We find that $u_0(t)$ satisfies

(2) $$u_0''(t) = \frac{h_0(t)}{2}, \quad u_0(0) = \frac{A_0}{2}, u_0'(0) = \frac{B_0}{2}.$$

A general solution has the form

$$u_0(t) = \int_0^t \left(\int_0^z \frac{h_0(s)}{2} ds\right) dz + \alpha t + \beta.$$

Comparing with (2), we find that $\beta = A_0/2$ and $\alpha = b_0/2 = B_0/2$. Thus,

$$u_0(t) = \int_0^t \left(\int_0^z \frac{h_0(s)}{2} ds \right) dz + \frac{A_0}{2} + \frac{B_0 t}{2}.$$

Note that in the homogeneous case, $h_0(s) = 0$, and so $u_0(t) = A_0/2 + B_0 t/2$.

We next consider $n \geq 1$.

Equating the Fourier coefficients of $\cos(\pi n x/L)$ in (1), we find that $h_n(t)$ satisfies

$$u_n''(t) + \left(\frac{\pi n c}{L}\right)^2 u_n(t) = h_n(t), u_n(0) = A_n, u_n'(0) = b_n. \tag{3}$$

The solution $u_n(t)$ can be found by method of variation of parameters, similarly to the case of Inhomogeneous wave equation with Dirichlet boundary conditions, discussed in the 2nd handout on wave equation.

Some important special cases, as well as examples, will be discussed in the next handout.