This is a summary of various results about solving inhomogeneous wave equations on the interval. For details, see §10.2 in the notes of Prof. Xu.

1. Inhomogeneous equation

1.1. Dirichlet BC. We would like to solve IBVP

(1) $$u_{tt} = c^2 u_{xx} + H(x,t), \quad u(x,0) = f(x), u_t(x,0) = g(x) \quad u(0,t) = 0 = u(L,t).$$

We look for solutions of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{\pi n x}{L}\right).$$

To find $u_n(t)$, we first expand $H(x,t)$ in Fourier series in x, where the coefficients are functions of t:

$$H(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{\pi n x}{L}\right),$$

where

$$h_n(t) = \frac{2}{L}\int_0^L H(x,t)\sin\left(\frac{\pi n x}{L}\right) dx.$$

We next expand $f(x)$ in sine Fourier series as usual,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{L}\right), \qquad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right) dx.$$

We also expand $g(x)$ in sine Fourier series,

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right), \qquad b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right) dx.$$

Let

$$B_n = \frac{b_n L}{\pi n c} = \frac{2}{\pi n c}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right) dx.$$

Equating the Fourier coefficients of $\sin(\pi n x/L)$ in (1), we find that $h_n(t)$ satisfies

(2) $$u_n''(t) + \left(\frac{\pi n c}{L}\right)^2 u_n(t) = h_n(t), u_n(0) = A_n, u_n'(0) = b_n.$$

The solution $u_n(t)$ can be found by method of variation of parameters; it is equal to

$$A_n \cos\left(\frac{\pi n c t}{L}\right) + B_n \sin\left(\frac{\pi n c t}{L}\right) + \frac{L}{\pi n c}\int_0^t h_n(s)\sin\left[\frac{\pi n c(t-s)}{L}\right] ds.$$

The solution $u(x,t)$ of (1) is then given by the formula

$$\sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{\pi nct}{L}\right) + B_n \sin\left(\frac{\pi nct}{L}\right) + \frac{L}{\pi nc} \int_0^t h_n(s) \sin\left[\frac{\pi nc(t-s)}{L}\right] ds \right\} \sin\left(\frac{\pi nx}{L}\right).$$