This is a summary of various results about solving constant coefficients wave equations on the interval, both homogeneous and inhomogeneous, with zero boundary conditions. For details, see §7.1 and §10.2 in the notes of Prof. Xu.

## 1. Homogeneous equation

We only give a summary of the methods in this case; details can be found in §7.1 of the notes of Prof. Xu.

### 1.1. **Dirichlet BC.** We would like to solve IBVP

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), u_t(x,0) = g(x) \quad u(0,t) = 0 = u(L,t). \tag{1}$$

To solve, expand $f(x)$ in *sine* Fourier series. The expansion is

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{L}\right), \qquad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right) dx.$$

Next, expand $g(x)$ in *sine* Fourier series. The expansion is

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right), \qquad b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right) dx.$$

Let

$$B_n = \frac{b_n L}{\pi n c} = \frac{2}{\pi n c}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right) dx.$$

The solution $u(x,t)$ of (1) is then given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{\pi n c t}{L}\right) + B_n \sin\left(\frac{\pi n c t}{L}\right)\right] \sin\left(\frac{\pi n x}{L}\right).$$

### 1.2. **Neumann BC.** We would like to solve IBVP

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), u_t(x,0) = g(x) \quad u_x(0,t) = 0 = u_x(L,t). \tag{2}$$

To solve, expand $f(x)$ in *cosine* Fourier series. The expansion is

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{L}\right), \quad A_0 = \frac{2}{L}\int_0^L f(x) dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right) dx.$$

Next, expand $g(x)$ in *cosine* Fourier series. The expansion is

$$g(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{\pi n x}{L}\right), \quad b_0 = \frac{2}{L}\int_0^L g(x) dx, \quad b_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{\pi n x}{L}\right) dx.$$

Let $B_0 = b_0$, and let

$$B_n = \frac{b_n L}{\pi n c} = \frac{2}{\pi n c} \int_0^L g(x) \cos\left(\frac{\pi n x}{L}\right) dx.$$

The solution $u(x,t)$ of (2) is then given by

$$u(x,t) = \frac{A_0}{2} + \frac{B_0 t}{2} + \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{\pi n c t}{L}\right) + B_n \sin\left(\frac{\pi n c t}{L}\right) \right] \cos\left(\frac{\pi n x}{L}\right).$$