Math 264: Advanced Calculus **Winter 2007**

Midterm Solutions

Please, solve all 4 problems. Each problem is worth 10 points.

Problem 1. Sketch the region S on the xy-plane bounded by the lines $x = 2$, $y = 1$ and the parabola $y = x^2$, and evaluate the double integral

$$\iint_S (x^2 + y^3)dxdy.$$

Solution. The region S has 3 "vertices": the points $(1, 1), (2, 1)$ and $(2, 4)$. The integral is equal to

$$\int_{x=1}^{2} \int_{y=1}^{x^2} (x^2 + y^3)dydx = \int_{x=1}^{2} \left[x^2(x^2 - 1) + \left(\frac{y^4}{4} \right)_{y=1}^{x^2} \right] dx =$$

$$\int_1^2 \left[x^4 + \frac{x^8}{4} - x^2 - \frac{1}{4} \right] dx = \left(\frac{x^5}{5} + \frac{x^9}{36} - \frac{x^3}{3} \right)_{x=1}^{2} - \frac{1}{4} = \frac{1603}{90}.$$

Problem 2. Compute

$$\iiint_R (\frac{x^2}{4} + \frac{y^2}{9} + z)dV,$$

where R is the elliptic cylinder $\{(x, y, z) : x^2/4 + y^2/9 \leq 1, -2 \leq z \leq 1\}$.
Solution. Change variables $x = 2u, y = 3v$. In the new coordinates, the integral I is equal to

$$6 \int_{z=-2}^{1} \int_{u^2+v^2 \leq 1, -2 \leq z \leq 1} (u^2 + v^2 + z)dudvdz.$$

Further changing to polar coordinates, $u = r\cos\theta, v = r\sin\theta$, we find that

$$I = 6 \int_{z=-2}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2 + z)rdrd\theta dz = 12\pi \int_{z=-2}^{1} \left(\frac{r^4}{4} + \frac{zr^2}{2} \right)_{r=0}^{1} dz$$

The last integral is equal to

$$12\pi \int_{z=-2}^{1} (1/4 + z/2)dz = 12\pi(3/4 + (z^2/4)_{-2}^{1}) = 0.$$

Problem 3. The surface S is the union of the disk $x^2 + y^2 \leq 9, z = 0$, and the "upper hemisphere" $x^2 + y^2 + z^2 = 9, z \geq 0$.

i) Find the area of S.
ii) Compute the outward flux of the vector field $\vec{\mathbf{F}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$ across S.

Solution. Part i) The area of S is equal to the sum of the area of the bottom disk $S_1 = \{x^2+y^2 \leq 9, z=0\}$ of radius 3 (and thus is equal to 9π); and the area of the upper hemisphere $S_2 = \{x^2+y^2+z^2 = 9, z \geq 0\}$.

S_2 can be parametrized by $(3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi)$, where $\theta \in [0, 2\pi], \phi \in [0, \pi/2]$. The tangent vectors are $T_\phi = (3\cos\phi\cos\theta, 3\cos\phi\sin\theta, -3\sin\phi)$ and $T_\theta = (-3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0)$. We have:

$$\mathbf{N} = T_\phi \times T_\theta = 9(\sin^2\phi\sin\theta, \sin^2\phi\cos\theta, \sin\phi\cos\phi).$$

We remark for part (ii) that since $\sin\phi\cos\phi \geq 0$ for $\phi \in [0, \pi/2]$, $\mathbf{N}$ is the *outward*-pointing normal. We find that $||\mathbf{N}|| = 9\sin\phi$.

The area of S_2 is equal to

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} ||T_\phi \times T_\theta|| d\theta d\phi = 18\pi \int_{\phi=0}^{\pi/2} \sin\phi d\phi = 18\pi(-cos\phi)_0^{\pi/2} = 18\pi.$$

Thus, the total area is equal to $9\pi + 18\pi = 27\pi$.

Solution. Part ii) The flux through S is equal to the sum of the flux through S_1 and S_2. We first compute the flux through S_2. Vector field $\vec{\mathbf{F}} = 3(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, and

$$\vec{\mathbf{F}} \cdot \mathbf{N} = 27(\sin^3\phi(\cos^2\theta + \sin^2\theta) + \sin\phi\cos^2\phi) = 27\sin\phi.$$

It follows that the flux through S_2 is equal to

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} (\vec{\mathbf{F}} \cdot \mathbf{N}) d\theta d\phi = 54\pi \int_{\phi=0}^{\pi/2} \sin\phi d\phi = 54\pi.$$

We can compute the flux through S_1 in two ways. We can remark that the outward pointing normal to S_1 is equal to $\mathbf{N} = (0, 0, -1)$: parametrize S_1 by $z = 0 = f(x, y)$ and use the fact that $N = (\partial z/\partial x, \partial z/\partial y, -1)$. Therefore, $\vec{\mathbf{F}} \cdot \mathbf{N} = (x, y, z) \cdot (0, 0, -1) = -z = 0$ on S_1. Integrating, we find that the flux is equal to zero.

Alternatively, we can parametrize S_1 by $(r\cos\theta, r\sin\theta, 0)$, where $r \in [0, 3], \theta \in [0, 2\pi]$. We find that $T_r = (\cos\theta, \sin\theta, 0)$, $T_\theta = (-r\sin\theta, r\cos\theta, 0)$, and $N = T_\theta \times T_r = (0, 0, -r)$ (this corresponds to the outward-pointing normal). Next, $\vec{\mathbf{F}} = (r\cos\theta, r\sin\theta, 0)$, and hence $\vec{\mathbf{F}} \cdot \mathbf{N} = 0$ and the flux is zero.

The total flux is thus equal to $54\pi + 0 = 54\pi$.

Problem 4. Consider the vector field

$$\vec{\mathbf{F}}(x, y, z) = (1+x)e^{x+y}\vec{\mathbf{i}} + (xe^{x+y} + 2y)\vec{\mathbf{j}} - 2z\vec{\mathbf{k}}.$$

i) Show that $\vec{F}$ is conservative by finding a potential for it.

ii) Evaluate

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}},$$

where $\mathcal{C}$ is given by

$$\vec{\mathbf{r}}(t) = (1+t)e^t\vec{\mathbf{i}} + 2t\vec{\mathbf{j}} + t\vec{\mathbf{k}}, \qquad (0 \leq t \leq 1).$$

Solution. The potential is equal to $F(x, y, z) = xe^{x+y} + y^2 - z^2$. The integral is equal to $F(2e, 2, 1) - F(1, 0, 0) = 2e^{3+2e} + 3 - e$