**Math 264: Advanced Calculus** **Winter 2007**

**Midterm Solutions**

Please, solve all 4 problems. Each problem is worth 10 points.

**Problem 1.** Sketch the region $S$ on the $xy$-plane bounded by the lines $x=2$, $y=1$ and the parabola $y=x^2$, and evaluate the double integral

$$\iint_S (x^3+y^2)dxdy.$$

**Solution.** The region $S$ has 3 "vertices": the points $(1,1),(2,1)$ and $(2,4)$. The integral is equal to

$$\int_{x=1}^{2}\int_{y=1}^{x^2}(x^3+y^2)dydx = \int_{x=1}^{2}\left[x^3(x^2-1)+\left(\frac{y^3}{3}\right)_{y=1}^{x^2}\right]dx =$$

$$\int_1^2\left[x^5+\frac{x^6}{3}-x^3-\frac{1}{3}\right]dx = \left(\frac{x^6}{6}-\frac{x^4}{4}+\frac{x^7}{21}\right)_{x=1}^{2}-\frac{1}{3}=\frac{349}{28}.$$

**Problem 2.** Compute

$$\iiint_R (\frac{x^2}{4}+\frac{y^2}{9}+z)dV,$$

where $R$ is the elliptic cylinder $\{(x,y,z): x^2/4+y^2/9\le 1, -1\le z\le 2\}$.
**Solution.** Change variables $x=2u, y=3v$. In the new coordinates, the integral $I$ is equal to

$$6\int_{z=-1}^{2}\int_{u^2+v^2\le 1,-1\le z\le 2}(u^2+v^2+z)dudvdz.$$

Further changing to polar coordinates, $u=r\cos\theta, v=r\sin\theta$, we find that

$$I=6\int_{z=-1}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r^2+z)rdrd\theta dz = 12\pi\int_{z=-1}^{2}\left(\frac{r^4}{4}+\frac{zr^2}{2}\right)_{r=0}^{1}dz$$

The last integral is equal to

$$12\pi\int_{z=-1}^{2}(1/4+z/2)dz = 12\pi(3/4+(z^2/4)_{-1}^{2}) = 18\pi.$$

**Problem 3.** The surface $S$ is the union of the disk $x^2+y^2\le 4, z=0$, and the "upper hemisphere" $x^2+y^2+z^2=4, z\ge 0$.

i) Find the area of $S$.
ii) Compute the outward flux of the vector field $\vec{\mathbf{F}}=x\vec{\mathbf{i}}+y\vec{\mathbf{j}}+z\vec{\mathbf{k}}$ across $S$.

**Solution. Part i)** The area of $S$ is equal to the sum of the area of the bottom disk $S_1 = \{x^2+y^2 \leq 4, z=0\}$ of radius 2 (and thus is equal to $4\pi$); and the area of the upper hemisphere $S_2 = \{x^2+y^2+z^2=4, z \geq 0\}$.

$S_2$ can be parametrized by $(2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, where $\theta \in [0, 2\pi], \phi \in [0, \pi/2]$. The tangent vectors are $T_\phi = (2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi)$ and $T_\theta = (-2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0)$. We have:

$$\mathbf{N} = T_\phi \times T_\theta = 4(\sin^2\phi\sin\theta, \sin^2\phi\cos\theta, \sin\phi\cos\phi).$$

We remark for part (ii) that since $\sin\phi\cos\phi \geq 0$ for $\phi \in [0, \pi/2]$, $\mathbf{N}$ is the *outward*-pointing normal. We find that $||\mathbf{N}|| = 4\sin\phi$.

The area of $S_2$ is equal to

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} ||T_\phi \times T_\theta|| d\theta d\phi = 8\pi \int_{\phi=0}^{\pi/2} \sin\phi d\phi = 8\pi(-cos\phi)_0^{\pi/2} = 8\pi.$$

Thus, the total area is equal to $4\pi + 8\pi = 12\pi$.

**Solution. Part ii)** The flux through $S$ is equal to the sum of the flux through $S_1$ and $S_2$. We first compute the flux through $S_2$. Vector field $\vec{\mathbf{F}} = 2(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, and

$$\vec{\mathbf{F}} \cdot \mathbf{N} = 8(\sin^3\phi(\cos^2\theta + \sin^2\theta) + \sin\phi\cos^2\phi) = 8\sin\phi.$$

It follows that the flux through $S_2$ is equal to

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} (\vec{\mathbf{F}} \cdot \mathbf{N}) d\theta d\phi = 16\pi \int_{\phi=0}^{\pi/2} \sin\phi d\phi = 16\pi.$$

We can compute the flux through $S_1$ in two ways. We can remark that the outward pointing normal to $S_1$ is equal to $\mathbf{N} = (0, 0, -1)$: parametrize $S_1$ by $z = 0 = f(x, y)$ and use the fact that $N = (\partial z/\partial x, \partial z/\partial y, -1)$. Therefore, $\vec{\mathbf{F}} \cdot \mathbf{N} = (x, y, z) \cdot (0, 0, -1) = -z = 0$ on $S_1$. Integrating, we find that the flux is equal to zero.

Alternatively, we can parametrize $S_1$ by $(r\cos\theta, r\sin\theta, 0)$, where $r \in [0, 2], \theta \in [0, 2\pi]$. We find that $T_r = (\cos\theta, \sin\theta, 0)$, $T_\theta = (-r\sin\theta, r\cos\theta, 0)$, and $N = T_\theta \times T_r = (0, 0, -r)$ (this corresponds to the outward-pointing normal). Next, $\vec{\mathbf{F}} = (r\cos\theta, r\sin\theta, 0)$, and hence $\vec{\mathbf{F}} \cdot \mathbf{N} = 0$ and the flux is zero.

The total flux is thus equal to $16\pi + 0 = 16\pi$.

**Problem 4.** Consider the vector field

$$\vec{\mathbf{F}}(x, y, z) = (1+x)e^{x+y}\vec{\mathbf{i}} + (xe^{x+y} + 2y)\vec{\mathbf{j}} - 2z\vec{\mathbf{k}}.$$

i) Show that $\vec{F}$ is conservative by finding a potential for it.

ii) Evaluate

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}},$$

where $\mathcal{C}$ is given by

$$\vec{\mathbf{r}}(t) = (1-t)e^t\vec{\mathbf{i}} + t\vec{\mathbf{j}} + 2t\vec{\mathbf{k}}, \qquad (0 \leq t \leq 1).$$

**Solution.** The potential is equal to $F(x,y,z) = xe^{x+y} + y^2 - z^2$. The integral is equal to $F(0,1,2) - F(1,0,0) = -e - 3$.