Problem 1. Use a line integral to find the plane area enclosed by the curve $\mathcal{C}$: $\mathbf{r} = a\cos^3 t\,\mathbf{i} + b\sin^3 t\,\mathbf{j} \quad (0 \le t \le 2\pi)$.

Solution: We assume $a > b > 0$.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi}(xy' - yx')dt = \frac{3ab}{2}\int_0^{2\pi}(\sin^4 t\cos^2 t + \sin^2 t\cos^4 t)dt \\ &= \frac{3ab}{8}\int_0^{2\pi}\sin^2(2t)dt = \frac{3\pi ab}{8}. \end{aligned}$$

Problem 2. Compute the outward flux of the vector field

$$\mathbf{F} = (e^{\cos z} + 3xy^2, 1/(10+\sin x) + 3x^2y, \sin(e^y) + z^3)$$

across the surface S consisting of the cylinder $x^2 + y^2 = 4, -2 \le z \le 0$, capped at the bottom by the disk $D := \{(x,y,z) : z = -2, 0 \le x^2 + y^2 \le 4\}$, and capped at the top by the hemisphere $\{x^2 + y^2 + z^2 = 4, z \ge 0\}$.

Solution: Let R be the region in $\mathbf{R}^3$ bounded by the surface S. We use the Divergence Theorem. We have $\text{div}\mathbf{F} = 3(x^2 + y^2 + z^2)$, so that

$$\int\int_S \mathbf{F} \bullet d\mathbf{S} = 3\int\int\int_D (x^2 + y^2 + z^2)dV := I.$$

To compute the integral, we decompose the domain D into 2 regions: $D = D_1 \cup D_2$, where

$$D_1 = \{(x,y,z) : -2 \le z \le 0, 0 \le x^2 + y^2 \le 4\},$$

and where

$$D_2 = \{(x,y,z) : z \ge 0, 0 \le x^2 + y^2 + z^2 \le 4\}.$$

Let $I_1 = 3\int\int\int_{D_1}(x^2 + y^2 + z^2)dV$. To compute I_1 we switch to cylindrical coordinates. The integral I_1 can be computed as follows:

$$I_1 = 3\int_{z=-2}^{0}\int_{r=0}^{2}\int_{\theta=0}^{2\pi}(r^2 + z^2)rdrd\theta dz = 6\pi\int_{z=-2}^{0}dz\left(\frac{r^4}{4} + \frac{r^2z^2}{2}\right)_0^2 = 80\pi.$$

Let $I_2 = 3\int\int\int_{D_2}(x^2 + y^2 + z^2)dV$. To compute I_2, we switch to spherical coordinates. We then have

$$I_2 = 3\int_{\rho=0}^{2}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi}\rho^2 \cdot \rho^2\sin\phi d\theta d\phi d\rho = 6\pi\,(-\cos\phi)_0^{\pi/2}\left(\frac{\rho^5}{5}\right)_0^2 = \frac{192\pi}{5}.$$

Finally, $I = I_1 + I_2 = 118.4\pi$.

Problem 3. Using Stokes's Theorem, compute the integral

$$\int_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz,$$

where C is the curve formed by the intersection of the cube $[0,1]\times[0,1]\times[0,1]\in\mathbf{R}^3$ with the plane $x+z=1$, oriented so that its projection into the (x,y)-plane has counterclockwise orientation.

Solution: We first remark that *curl* $\mathbf{F}=(-2)\cdot(y+z,x+z,x+y)$. The intersection S of the unit cube with the plane $x+z=1$ is a rectangle with vertices at $(1,0,0),(1,0,1),(0,1,0),(0,1,1)$. It projects one-to-one onto the unit square $[0,1]\times[0,1]$ in the (x,z)-plane, and the boundary curve is oriented counterclockwise. Accordingly, S can be parametrized by $\{(x,y,1-x):0\le x,y\le 1\}$. The normal $\mathbf{N}$ is pointing upward and is equal to $(1,0,1)$.

Thus, $(\nabla\times\mathbf{F})\bullet\mathbf{N}=(-2)(x+2y+z)=(-2)(2y+1)$. the flux integral is equal to

$$(-2)\int_{x=0}^{1}\int_{y=0}^{1}(2y+1)dydx=-4.$$

Problem 4. Let

$$\mathbf{F}=\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}+18xz^2,\frac{y}{(x^2+y^2+z^2)^{3/2}}+\frac{yx^2}{2}+\frac{2y^3}{3},\frac{z}{(x^2+y^2+z^2)^{3/2}}\right).$$

Compute the outward flux of $\mathbf{F}$ through the boundary of the ellipsoid

$$x^2/4+y^2+9z^2=1.$$

Solution: We write $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2$, where

$$\mathbf{F}_1=\left(\frac{x}{(x^2+y^2+z^2)^{3/2}},\frac{y}{(x^2+y^2+z^2)^{3/2}},\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)=\frac{\mathbf{r}}{r^3},$$

which has the singularity at $\mathbf{r}=0$, and

$$\mathbf{F}_2=(18xz^2,yx^2/2+2y^3/3,0).$$

Note that

$$\text{div}\mathbf{F}_1=0,$$

and

$$\text{div}\mathbf{F}_2=18z^2+x^2/2+2y^2.$$

Let S be the surface of the ellipse; then

$$\int\int_S\mathbf{F}\bullet d\mathbf{S}=\int\int_S\mathbf{F}_1\bullet d\mathbf{S}+\int\int_S\mathbf{F}_2\bullet d\mathbf{S}:=I_1+I_2.$$

Since the origin is contained in the ellipsoid R bounded by S, to compute I_1, by applying the divergence theorem, we may let (S_0) be a sphere with radius ϵ. Then,

$$I_1=\int\int_S\mathbf{F}_1\bullet d\mathbf{S}=\int\int_{(S_0)}\mathbf{F}_1\bullet d\mathbf{S}=\int\int_{(S_0)}\frac{\mathbf{r}}{r^3}\bullet\frac{\mathbf{r}}{r}dS=\int\int_{(S_0)}\frac{1}{r^2}dS$$
$$=\int\int_{(S_0)}\frac{1}{\epsilon^2}dS=4\pi.$$

To compute I_2, we again apply the Divergence Theorem. We have $\text{div}\mathbf{F}_2 = 18z^2 + x^2/2 + 2y^2$. Then

$$I_2 = \int\int\int_{x^2/4+y^2+9z^2\leq 1} (x^2/2 + 2y^2 + 18z^2)dxdydz.$$

We change coordinates: $u = x/2, v = y, w = 3z$; the Jacobian is equal to $2/3$. The integral becomes

$$\frac{2}{3}\int\int\int_{u^2+v^2+w^2\leq 1} 2(u^2 + v^2 + w^2)dudvdw.$$

Switching to spherical coordinates, we find that

$$I_2 = \frac{4}{3}\int_{\rho=0}^{1}\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \rho^2 \cdot \rho^2 \sin\phi d\rho d\phi d\theta = \frac{8\pi}{3}(-\cos\phi)_0^{\pi}(\rho^5/5)_0^1 = \frac{16\pi}{15}.$$

The total flux is equal to $I_1 + I_2 = 4\pi + 16\pi/15 = 76\pi/15$.

Problem 5. Given he following IBVP of heat conduction equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 3\sin x \quad (0 < x < \pi, t > 0)$$
$$u(0,t) = 1, u(\pi,t) = 0, \quad (t > 0)$$
$$u(x,0) = 2\sin(3x) - \sin(5x), \quad (0 \leq x \leq \pi)$$

(1) Find the equilibrium state solution for the above problem;
(2) Find the unsteady solution for the above IBVP;
(3) Show that the above unsteady solutions approach to the equilibrium state solution as $t \to \infty$.

Solution: (1).We look for steady solution of the inhomogeneous equation in the form $u(x,t) = W(x)$, where $d^2W/dx^2 = 3\sin x, W(0) = 1, W(\pi) = 0$. By integration, we find that $W(x) = 1 - \frac{x}{\pi} - 3\sin x$.

(2). Let $u(x,t) = \tilde{u}(x,t) + W(x)$. Substituting into the equation, we find that $\tilde{u}$ satisfies

$$\tilde{u}_t = \tilde{u}_{xx};$$
$$\tilde{u}(0,t) = \tilde{u}(\pi,t) = 0;$$
$$\tilde{u}(x,0) = -1 + \tfrac{x}{\pi} + 3\sin x + 2\sin(3x) - \sin(5x) = f(x).$$

To solve $\tilde{u}$, we use the Sine Fourier series:

$$\tilde{u}(x,t) = 3\sin x\mathrm{e}^{-t} + 2\sin(3x)\mathrm{e}^{-3^2t} - \sin(5x)\mathrm{e}^{-5^2t} + \sum_{n=1}^{\infty}\tilde{a}_n \sin nx\mathrm{e}^{-n^2t}.$$

From the IC, we derive

$$\sum_{n=1}^{\infty}\tilde{a}_n \sin nx = \frac{x}{\pi} - 1, \quad (0 \leq x \leq \pi).$$

Thus, it follows that

$$\tilde{a}_n = \frac{2}{\pi}\int_0^\pi \left(\frac{x}{\pi} - 1\right)\sin nx\mathrm{d}x = -\frac{2}{n\pi}\cos nx\left(\frac{x}{\pi} - 1\right)_0^\pi = \frac{2(-1)^n}{n\pi}.$$

Therefore, we finally obtain:

$$u(x,t) = 1 - \tfrac{x}{\pi} - 3\sin x + 3\sin x\mathrm{e}^{-t} + 2\sin(3x)\mathrm{e}^{-3^2 t} - \sin(5x)\mathrm{e}^{-5^2 t}$$
$$+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin nx\mathrm{e}^{-n^2 t}.$$

(3). As $t \to \infty$, $u(x,t) \to W(x) = 1 - \frac{x}{\pi} - 3\sin x$.

Problem 6. Given the following problem of the string vibration with external oscillatory force:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \cos 2\pi x\cos 2\pi t, \quad (0 < x < 1, t > 0)\\
&u_x(0,t) = u_x(1,t) = 0, \quad (t > 0)\\
&u(x,0) = f(x) = \cos^2\pi x, \quad u_t(x,0) = g(x) = 2\cos 2\pi x, \quad (0 \le x \le 1).
\end{aligned}$$

(1) Use the method of separation of variables and Fourier series to find its solution;
(2) Determine the behavior of the solution, as $t \to \infty$: Is the solution bounded or unbounded as $t \to \infty$?

Solution: 1. We first look for a particular solution of the form $W(x,t) = \cos(2\pi x)\phi(t)$. Substituting into the equation we get

$$\cos(2\pi x)\phi''(t) + 4\pi^2\cos(2\pi x)\phi(t) = \cos(2\pi x)\cos(2\pi t).$$

It follows that $\phi''(t) + 4\pi^2\phi(t) = \cos(2\pi t)$. From the theory of inhomogeneous second order ODE (method of undetermined coefficients), we know that solution will have the form $\phi(t) = A\cos(2\pi t) + B\sin(2\pi t) + Ct\cos(2\pi t) + Dt\sin(2\pi t)$. Substituting into the ODE, we find that Taking the second derivative and equating the coefficients, we find that

$$\phi(t) = t\sin(2\pi t)/(4\pi), \quad W(x,t) = t\sin(2\pi t)/(4\pi)\cos(2\pi x).$$

We now set $u(x,t) = \tilde{u}(x,t) + W(x,t)$. Then by substituting into the equation we have

$$\begin{aligned}
&\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} = 0, \quad (0 < x < 1, t > 0)\\
&\tilde{u}_x(0,t) = \tilde{u}_x(1,t) = 0, \quad (t > 0)\\
&\tilde{u}(x,0) = f(x) = \cos^2\pi x, \quad \tilde{u}_t(x,0) = g(x) = 2\cos 2\pi x, \quad (0 \le x \le 1).
\end{aligned}$$

One may make Cosine Fourier series for $\tilde{u}(x,t)$,

$$\tilde{u}(x,t) = \sum_{n=0}^{\infty} \Big[A_n \cos(n\pi t) + B_n \sin(n\pi t)\Big] \cos(n\pi x).$$

To satisfy the IC's, we have

$$\tilde{u}(x,0) = \sum_{n=0}^{\infty} A_n \cos(n\pi x) = f(x) = \frac{1+\cos(2\pi x)}{2}.$$

and

$$\tilde{u}_t(x,0) = \sum_{n=0}^{\infty} 2\pi B_n \cos(n\pi x) = g(x) = 2\cos(2\pi x).$$

It follows that

$$A_0 = A_2 = \frac{1}{2}, \quad B_2 = \frac{1}{\pi}, \quad A_n = B_n = 0, \quad \text{otherwise.}$$

Hence,

$$\tilde{u}(x,t) = \frac{1}{2} + \left[\frac{1}{2}\cos(2\pi t) + \frac{1}{\pi}\sin(2\pi t)\right]\cos(2\pi x).$$

Finally, we get

$$u(x,t) = \frac{1}{2} + \left[\frac{1}{2}\cos(2\pi t) + \frac{1}{\pi}\sin(2\pi t)\right]\cos(2\pi x) + \frac{t}{4\pi}\sin(2\pi t)\cos(2\pi x).$$