Math 264: Advanced Calculus **Winter 2007**

FINAL EXAM **Solutions**

Problem 1. Compute

$$\int_C \left(x + y + \frac{-y}{x^2 + y^2} \right) dx + \left(y - x + \frac{x}{x^2 + y^2} \right) dy,$$

where C is the ellipse $x^2/16 + 9y^2 = 1$.

Solution. We have to compute $\int_C \mathbf{F} \cdot \mathbf{ds}$, where $\mathbf{F} = \left(x + y + \frac{-y}{x^2+y^2}, y - x + \frac{x}{x^2+y^2} \right)$. Write $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where $\mathbf{F}_1 = (x + y, y - x)$ and $\mathbf{F}_2 = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$. By a result proved in class, $\int_C \mathbf{F}_2 \cdot \mathbf{ds} = 2\pi$, since the origin $(0, 0)$ lies inside the region R enclosed by the ellipse. To compute $\int_C \mathbf{F}_1 \cdot \mathbf{ds}$, we use Green's theorem. We have $\mathbf{F}_1 = (P, Q) = (x + y, y - x)$, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2.$$

Accordingly, if we denote by E the ellipse $\{x^2/16 + 9y^2 \leq 1\}$, we have

$$\int_C \mathbf{F}_1 \cdot \mathbf{ds} = \iint_E (-2) dx dy = (-2)\text{Area}(E) = -2\pi \cdot 4/3 = -8\pi/3.$$

Here we have used the formula $\text{Area}(A) = \pi ab$, where $a = 4$ and $b = 1/3$ are semiaxes of the ellipse.

The final answer is equal to $2\pi - 8\pi/3 = -2\pi/3$.

Problem 2. Compute the outward flux of the vector field $\mathbf{F} = (y^2x - xz, yz + x^2y, e^x + \cos(y))$ across the surface S consisting of the paraboloid $z = x^2 + y^2, 0 \leq z \leq 4$, capped by the disk $D := \{(x, y, z) : z = 4, 0 \leq x^2 + y^2 \leq 4\}$.

Solution. To compute the flux, we use the Divergence theorem. We have

$$\text{div}\mathbf{F} = \frac{\partial(y^2x - xz)}{\partial x} + \frac{\partial(yz + x^2y)}{\partial y} + \frac{\partial(e^x + \cos(y))}{\partial z} = y^2 - z + z + x^2 = x^2 + y^2.$$

Accordingly, if we denote by R the region bounded by S, we have

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iiint_R (x^2 + y^2) dV.$$

To compute the last integral, we use cylindrical coordinates. Then $x^2 + y^2 = r^2$, and the integral becomes

$$\int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{z=r^2}^{4} r^2 r dr d\theta dz = 2\pi \int_{r=0}^{2} r^3(4 - r^2) dr = 2\pi(r^4 - r^6/6)_0^2 = \frac{32\pi}{3}.$$

Problem 3. Using Stokes' Theorem, compute the integral $\int_C (x^2 - yz)dx + (2x + y^2 - xz)dy + (z^2 - xy)dz$, where C is the curve formed by the intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the plane $z = 4$, oriented counterclockwise (e.g. its projection into the (x, y)-plane is oriented counterclockwise).

Solution. The plane $z = 4$ intersects the sphere in a disk $D = \{z = 4, 0 \leq x^2 + y^2 \leq 9\}$. We have

$$\int_C \mathbf{F} \cdot \mathbf{ds} = \iint_D (\text{curl}\mathbf{F}) \cdot \mathbf{dS} = \iint_D (\text{curl}\mathbf{F}) \cdot \mathbf{n} dA.$$

We next find that

$$\text{curl}\mathbf{F} = (0, 0, 2).$$

The unit $\mathbf{n}$ normal to the surface D is equal to $\pm(0, 0, 1)$. The curve C is oriented so that we choose the $+$ sign. Therefore, $\mathbf{F} \cdot \mathbf{n} = 2$. Accordingly, the integral is equal to

$$\int_D (\text{curl}\mathbf{F}) \cdot \mathbf{n} dA = 2 \cdot \text{Area}(D) = 2\pi \cdot 3^2 = 18\pi.$$

Problem 4. Compute the surface integral

$$\iint_S (x^2 + y^2 + z^2) dA,$$

where S is the surface of the tetrahedron with vertices at

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1).$$

Solution. Let $A = (0, 0, 0), B = (1, 0, 0), C = (0, 1, 0), D = (0, 0, 1)$. By symmetry, the integral (call it I) that we need to compute satisfies $I = 3 \iint_{ABC} f \; dS + \iint_{BCD} f \; dS$, where $f(x, y, z) = x^2 + y^2 + z^2$.

Now, the triangle ABC lies in the xy-plane, so $z = 0$ and the integral $I_1 = \int_{ABC} f \; dS$ satisfies

$$\begin{aligned} I_1 &= \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2 + y^2) dy dx = \int_{x=0}^{1} \left(x^2 y + \frac{y^3}{3} \right)_{y=0}^{1-x} dx \\ &= \int_0^1 \left(\frac{1}{3} - x + 2x^2 - \frac{4x^3}{3} \right) dx = \frac{1}{6}. \end{aligned}$$

Next, the triangle BCD lies in the plane $x + y + z = 1$ or $z = 1 - x - y$. The normal N to the plane is a vector $1, 1, 1$, and the area form is $\sqrt{3} dx dy$. Accordingly, the integral $I_2 = \int_{BCD} f \, dS$ is equal to

$$
\begin{aligned}
I_2 &= \sqrt{3} \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2 + y^2 + (1-x-y)^2) dy dx \\
&= \sqrt{3} \int_{x=0}^{1} \left(2x^2 y + \frac{2y^3}{3} + xy^2 - 2xy - y^2 + y \right)_{y=0}^{1-x} dx \\
&= \sqrt{3} \int_{0}^{1} \left(\frac{2}{3} - 2x + 3x^2 - \frac{5x^3}{3} \right) dx = \frac{\sqrt{3}}{4}.
\end{aligned}
$$

The final answer is equal to $3I_1 + I_2 = 1/2 + \sqrt{3}/4$.

Problem 5. Use separation of variables method to solve the heat equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad 0 < t < \infty, \\
u(0,t) &= u(\pi,t) = 0, && 0 < t < \infty, \\
u(x,0) &= \sin(x) - 6\sin(4x), && 0 < x < \pi.
\end{aligned}
$$

Solution: This is a heat problem with homogeneous Dirichlet boundary conditions. Using separation of variables method the solution takes the form

$$
u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right),
$$

where $\beta = 3$ and $L = \pi$. Then

$$
u(x,t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin(nx).
$$

By the initial condition, we find

$$
u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = \sin(x) - 6\sin(4x)
$$

Comparing coefficients, we get

$$
c_1 = 1, \qquad c_4 = -6,
$$

and the remaining $c_n's$ are zero. Therefore, the solution of the heat problem is

$$
u(x,t) = e^{-3t}\sin(x) - 6e^{-48t}\sin(4x)
$$

Problem 6. Use Fourier series to solve the heat equation

$$\begin{aligned}
\frac{\partial u}{\partial t} &= 7\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad 0 < t < \infty, \\
\frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(\pi,t) = 0, && 0 < t < \infty, \\
u(x,0) &= 1 - \sin(x), && 0 < x < \pi.
\end{aligned}$$

Solution : This is a heat problem with homogeneous Neumann boundary conditions. Thus the solution has the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

with $\beta = 7$ and $L = \pi$, hence

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-7n^2 t} \cos(nx)$$

The coefficients of the Fourier cosine series are given by

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi (1-\sin(x))\cos(nx)dx = \frac{2}{\pi}\int_0^\pi \cos(nx)dx - \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)dx \\
&= \frac{2}{\pi}\frac{\sin(nx)}{n}\bigg|_0^\pi - \frac{2}{\pi}\left[-\frac{1}{2}\frac{\cos((n+1)x)}{n+1} + \frac{1}{2}\frac{\cos((n-1)x)}{n-1}\right]\bigg|_0^\pi, \quad n \neq 0,1 \\
&= \frac{1}{\pi}\left[\frac{\cos((n+1)x)}{n+1} - \frac{\cos((n-1)x)}{n-1}\right]\bigg|_0^\pi, \quad n \neq 0,1 \\
&= \frac{1}{\pi}\left[\left(\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right) - \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right] \quad n \neq 0,1 \\
&= \frac{2}{\pi}\frac{n((-1)^n+1)}{n^2-1} = \begin{cases} 0, & n \;\; odd \\ \dfrac{4n}{(n^2-1)\pi}, & n \;\; even \end{cases}
\end{aligned}$$

If $n = 0$, we obtain

$$a_0 = \frac{2}{\pi}\int_0^\pi (1-\sin(x))dx = \frac{2}{\pi}(x+\cos(x))\bigg|_0^\pi = \frac{2(\pi-2)}{\pi}$$

If $n = 1$, we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi}(1-\sin(x))\cos(x)dx = \frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(x)dx = \frac{1}{\pi}\int_0^{\pi}\sin(2x)dx = 0$$

Therefore, the solution is

$$\begin{aligned} u(x,t) &= \frac{\pi-2}{\pi} + \sum_{n=2,4,..}\frac{4n}{(n^2-1)\pi}e^{-7n^2t}\cos(nx) \\ &= \frac{\pi-2}{\pi} + \sum_{n=1}^{\infty}\frac{8n}{(4n^2-1)\pi}e^{-28n^2t}\cos(2nx) \end{aligned}$$